Log rules

$a^{k\log b} = b^{k\log a}$

$\log a^b = b\log a$

$a^{\log_a b} = b$

$\log_a b = \frac{\log_c b}{\log_c a}$

Bound notations

$f(n)$: o, O, Θ, Ω, ω

Recurrence banks

$T(n) =$

$T(\frac{n}{2}) + \log n \Rightarrow O(\log^2 n)$

$2T(n-1) + n \Rightarrow O(2^n)$

$3T(\frac{n}{2}) + n^{\frac{1}{2}} \Rightarrow O(n^{\log_2 3})$

$T = T(\sqrt[r]{n}) + \cdots$

$\underbrace{(\cdots(r^r)^r\cdots)^r}_{d} = n$

$r^{r^d} = n$

$r^d = \log_r n$

depth: $d = \log_r \log_r n$

$T(n) = cT(\frac{n}{d}) + \cdots$

depth: $\log_d n$

leaves: $n^{\log_d c}$

Wacky Tree

$W(n) = W(\frac{n}{2}) + W(\frac{n}{3}) + 1$

Leaf Guess $L(n) = n^\beta$

$L(n) = L(\frac{n}{2}) + L(\frac{n}{3})$

$n^\beta = (\frac{n}{3})^\beta + (\frac{n}{3})^\beta$

Solve $\Rightarrow$ $\beta \approx 0.788$.

$O(n^{0.788})$

Summations

$\sum_{i=0}^{n} i^a \in \Theta(n^{a+1})$

$\sum_{i=0}^{n} \frac{1}{i} \in \Theta(\log n)$

$\sum_{i=0}^{n} b^i \in \Theta(b^n)$

$\sum_{i=0}^{\log n} \frac{n}{2^i} \in \Theta(n)$

$\sum_{i=0}^{n} (\log^c i)(i^a)(b^i)$ $b \geq 1$, $a, c \geq 0$

$\in \Theta\left((n\log^c n)(n^a)(b^n)\right)$

Quicksort

X_{ij} indicate i,j compared, $i<j$

$j-i+1$, i, j

$\mathbb{E}[X_{ij}] = \frac{1}{j-i+1}(2!)$ order to pick i,j

$\mathbb{E}[\#\text{comparisons}] = \sum_{i<j}\mathbb{E}[X_{ij}] \leq 2\sum_{i=0}^{n} H_n$

$\mathbb{P}[\text{one path} > k\lg n] \leq \frac{1}{n^k}$

Prob

Random var $X : \Omega \to \mathbb{R}$

Expected var $\mathbb{E}[X]$ = weighted sum

Indep X, Y indep $\Leftrightarrow$ $\mathbb{P}[X=a, Y=b] = \mathbb{P}[X=a]\mathbb{P}[Y=b]$

Linearity $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Product expectation $\mathbb{E}[X\cdot Y] = \mathbb{E}[X]\cdot\mathbb{E}[Y]$

Union bound $P(A\cup B) \leq P(A) + P(B)$

Conditional $P(A|B) = \frac{P(A\cap B)}{P(B)}$

Markov inequality $X \geq 0 \Rightarrow \mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$

Prob bound

High Prob Bound $W(n) \in O(f(n))$ w.h.p. if $\exists$ constants c, n_0 s.t. $\forall n > n_0, \forall k, \mathbb{P}[W(n) \leq ckf(n)] \geq 1 - (\frac{1}{n})^k$

Event A has high prob scaled with n if $\mathbb{P}[\neg A] \leq \frac{1}{n^k}$

Proving
1. N_r := input size at round r
2. Decay ratio $\frac{\mathbb{E}[N_r]}{\mathbb{E}[N_{r-1}]} \leq p$
3. $\mathbb{E}[N_r] \leq p^r N_0 = p^r n$
4. Set $r = (k+1)\log_{\frac{1}{p}} n$, show $\mathbb{E}[N_r] \leq \frac{1}{n^k}$

Partition cost $\mathbb{E}[T(n)] = \mathbb{E}[T(n) \mid \text{good}]\mathbb{P}[\text{good}] + \cdots$

Nested parallel work-span model

min time bound, greedy scheduling bound

$\max(\frac{W}{P}, S) \leq T_P \leq \frac{W}{P} + S \leq 2\max(\frac{W}{P}, S)$

Want $\frac{W}{P}$ dominate $\Rightarrow S < \frac{W}{P}$

$\Leftrightarrow P < \left[\frac{W}{S}\right]$ parallelism

Work efficient := $W_{\text{impl}}^{\text{par}} \in O(W_{\text{seq impl}}^{\text{best}})$

Perf

T^* – sequential baseline

T_P – on P processors $T_P \geq \frac{T_1}{P}$

Self-speedup $= \frac{T_P}{T_1} \leq P$ └ perfect speedup if they equal

Overhead $= \frac{T_1}{T^*} \geq 1$

Efficient reduction := transformation takes no more asymptotic cost than solving transformed problem.

Misc

$H_n = \sum_{i=1}^{n}\frac{1}{i} \leq \ln n + O(1)$ $\|F\| = \sum_{x\in F}(d^+(x)+1)$

BST perfect balanced depth $= \lceil \lg(n+1) \rceil$

Inorder: inord L ++ ⟨x⟩ ++ inord R

$S_n = ar^0 + ar^1 + ar^2 + \cdots + ar^{n-1}$

$= a\left(\frac{1-r^n}{1-r}\right)$

Sequences

$\langle i : 0 \leq i < n \mid i\%2 = 0 \rangle$ map/tab, filter

$A[1\cdots4]$ inclusive

$A +\!\!+ B$ append

reduce f b S usually $W(n) = 2W(\frac{n}{2}) + W_f(n)$

scan f b S usually $W(n) = W(\frac{n}{2}) + W_{\text{contract}}(n) + W_{\text{expand}}(n)$

scan append $\langle\langle m\rangle, \langle m\rangle^n, \ldots, \langle m\rangle\rangle$

	contract	recur	expand	solution
W	$2m\cdot\frac{n}{2} \in O(mn)$	$W(\frac{n}{2}, 2m)$	$O(n^2 m)$	$O(n^2 m)$
S	$O(1)$	$S(\frac{n}{2}, 2m)$	$O(1)$	$O(\lg n)$

Sorting $W = O(n\log n)$
$S = O(\log^2 n)$

scan op+ 0 ⟨2 1 1 4 3 2 1 3⟩

contract ⟨3 5 5 4⟩

recur (⟨0 3 8 13⟩, 17)

expand (⟨0 2 3 4 8 11 13 14⟩, 17)

Substitution

Recurrence

$W(n) = 5W(\frac{n}{8}) + O(n^{\frac{2}{3}})$

$\leq 5W(\frac{n}{8}) + c_1 n^{\frac{2}{3}}$

Guess Leaf … $O(n^{\log_8 5})$

Have $\mathbb{E}X_0 = n$, $\mathbb{E}X_{i+1} = (\frac{1}{2})^i n$

Fix k, WTS done in $O(k\lg n)$ whp.

$r := (k+1)\log_{(\frac{1}{2})^{-1}} n$

$\mathbb{P}[X_r \geq 1] \leq \mathbb{E}X_r$

$= (\frac{1}{2})^{(k+1)\lg n} n$

$= n^{(k+1)(-1)} n$

$= \frac{1}{n^k}$

Claim $W(n) \leq k_1 n^{\log_8 5} + k_2$

BC $W(1) \leq c_0 \leq k_1 + k_2$

IH $\forall_{1\leq n'<n}$ $W(n') \leq k_1 n'^{\log_8 5} + k_2$

IS $W(n) \leq 5W(\frac{n}{8}) + c_1 n^{\frac{2}{3}}$

$\leq 5(k_1(\frac{n}{8})^{\log_8 5} + k_2) + c_1 n^{\frac{2}{3}}$

$\vdots$

$= k_1 n^{\log_8 5} + 5k_2 + c_1 n^{\frac{2}{3}}$

$= k_1 n^{\log_8 5} + k_2 + 4k_2 + c_1 n^{\frac{2}{3}}$ uh oh

Claim' $W(n) \leq k_1 n^{\log_8 5} + k_2 + k_3 n^{\frac{2}{3}}$ BC' …

IH' $\forall_{1\leq n'<n}$ $W(n') \leq k_1 n'^{\log_8 5} + k_2 + k_3 n'^{\frac{2}{3}}$

IS' $W(n) \leq 5W(\frac{n}{8}) + c_1 n^{\frac{2}{3}}$

$\leq 5(k_1(\frac{n}{8})^{\log_8 5} + k_2 + k_3(\frac{n}{8})^{\frac{2}{3}}) + c_1 n^{\frac{2}{3}}$

$\vdots$

$= k_1 n^{\log_8 5} + k_2 + k_3 n^{\frac{2}{3}} + \underbrace{4k_2 + c_1 n^{\frac{2}{3}} + \frac{1}{4}k_3 n^{\frac{2}{3}}}_{\text{now make this} \leq 0}$

Define $k_2 = c_1$ $k_3 = -20$ $k_1 = 20$

mcss A = let
(b, v) = scan op+ 0 A
prefixSums = append b ⟨v⟩
(minPrefixes, _) = scan min ∞ prefixSum
maxForEnds =
⟨ prefixSums [i] − minPrefix [i] : 0 ≤ i ≤ |A| ⟩
in
reduce max −∞ maxForEnds
end

Treaps

$\mathbb{P}[A_j^i] = \frac{1}{|i-j|+1}$ (i ancestor of j) ← need choose i first

$\mathbb{E}[\text{depth}(i)] = \sum_{i=0}^{n-1}\mathbb{E}[A_j^i] \begin{cases}\leq 2H_n \\ \in O(\lg n) \text{ w.h.p}\end{cases}$

$\mathbb{E}[\text{size}(i)] \in O(\lg n)$ (not w.h.p.!)

treap height $\in O(\lg n)$ w.h.p.

Def balanced : height $\in O(\lg n)$

Dijkstra

Prop $p(v) := \min_{u \in X} (\delta(s,u) + w(u \to v))$

Then $v \in V \setminus X$ with $\min p(v)$ has $\delta(s,v) = p(v)$

viz. best way to expand seen set is greedily taking min way out

i.e. $\min_{v \in V \setminus X} p(v) = \min_{v \in V \setminus X} \delta(s,v)$

```
DijkstraPQ G s =
  let loop X Q =
    case delMin Q of
      (NONE, _) ⇒ X
     |(SOME (d,v), Q') ⇒
        if (v,_) ∈ X then loop X Q'
        else let
          X' = X ∪ {(v,d)}
          relax (Q, (u,w)) = insert (Q, (d+w, u))
          Q'' = iterate relax Q' N⁺_G(v)
        in
          loop X' Q''
        end
  in
    loop {} (insert emptyQ (0,s))
  end
```

Associativity

Identity for max $-\infty$

min ∞

Tree pq impl $O(m \lg n) = O(m \lg m)$

Dijkstra alt with Fibb heap < $O(1)$ insert, update; $O(\lg n)$ delMin

$W = O(m + n \lg n)$

Bellman-Ford

k-hop shortest, then relax every next hop. n-1 hops enough

```
BF G s =
  let loop (D : (V, R) table) k =
    let D' = {v ↦ min (D[v], min_{u ∈ N⁻_G(v)} D[u] + w(v,u)) : v ∈ V}
    in if k = |V| then NONE
       else if D' = D then SOME D
       else loop D' (k+1)
  in loop {s ↦ 0} ∪ {v ↦ ∞ : v ∈ V\{s}} 0
  end
```

min cost to hop into vert from end of prev hops

Treap Ops

WLOG $n \geq m$

	W	S
∩, ∪, \	$m \lg(1 + \frac{n}{m})$	$\lg(m+n)$
domain, range, toSeq	n	$\lg n$
collect, fromSeq	$n \lg n$	$\lg^2 n$
join, joinM	$\lg(m+n)$	$\lg(m+n)$
split, find, insert, delete	$\lg n$	$\lg n$

Topological Sort, SCC, Kosaraju

Def $a < b \Leftarrow b$ reachable from $a \wedge b \neq a$

decreasingFinish given DAG:

$\Sigma_0 = \langle \rangle$

visit $\Sigma\ v = \Sigma$

finish $\Sigma\ v = \langle v \rangle + \Sigma$

revisit $\Sigma\ v = \Sigma$

decreasingFinish G = #1 (DFSAll G Σ_0)

SCC G =   Cost ~ 2× DFS

```
SCC G =
  let
    F = revFinishTime G
    Gᵀ = transpose G
    accumSCCs ((X, L), u) =
      let
        (X', A) = reach Gᵀ X u
      in
        if A = ∅ then (X, L) else
        (X', L ++ ⟨A⟩)
  in
    iterate accumSCCs (∅, ⟨⟩) F
  end
```

X: visited verts; L: SCCs so far; X': overall visited; A: newly visited; reach: look for reachable, e.g. DFS

Lemma If $v \in V$ is first visited vert in its SCC, all verts u reachable from v finish before v.

Graph Search Summary

W, S, x = lg n if visit/revisit/finish are O(lg n)

	W	S
DFS	$m+n$	$m+n$ (revisit max n)
BFS (table)	$(m+n)\lg n$ (set op)	$d \lg^2 n$ (depth)
BFS (seq)	$m+n$	$d \lg n$
Dijkstra	$m \lg n$; $m + n\lg n$	$m \lg n$; $m + n\lg n$ ← with decreaseKey
BF (seq)	mn (update all node dist for every edge)	$n \lg n$; $O(\frac{n}{\lg n})$ parallelism
BF (table)	$mn \lg n$	$n \lg n$

Generic DFS

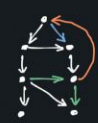
```
DFS G ((Σ, X), v) =
  if v ∈ X then (revisit (Σ, v), X)
  else let
    Σ' = visit (Σ, v)
    X' = X ∪ {v}
    (Σ'', X'') = iterate (DFS G) (Σ', X') N⁺(v)
  in
    (finish (Σ'', v), X'')
  end

DFSAll G Σ = iterate (DFS G) (Σ, ∅) V
```

Edge types

- Tree
- Back
- Forward
- Cross

in G, taken by DFS



$\exists$ Cycle $\Leftrightarrow \exists$ back edge in DFS

MST

Light edge prop let $U \subseteq V$ then lightest edge btwn U and $V \setminus U$ is in MST

Cycle prop for all cycle in G, the heaviest edge in the cycle is not in MST

MST ≈ TSP $W(MST) \leq W(TSP) \leq W(MST)$

	W	S	
Prim	$m \lg n$	$m \lg n$	greedy expand by lightest out
Kruskal	$m \lg n$	$m \lg n$	keep adding lightest connector edge
Borůvka (tree)	$m \lg n$	$\lg^3 n$	in parallel find min edge incident
Borůvka (star)	$m \lg n$	$\lg^2 n$	to every vert, contract them
Sleator-Tarjan	?	?	iteratively add edges, remove heaviest of cycle created

Path potential lemma

for weighted graph $G = (V,E,w)$ and potentials $\varphi(v) : V \to \mathbb{R}$

reweight by $w'(u,v) = w(u,v) + \varphi(u) - \varphi(v)$ into $G' = (V,E,w')$

we have $\delta_G(u,v) = \delta_{G'}(u,v) - \varphi(u) + \varphi(v)$

```
JohnsonAPSP (G = (V,E,w)) =
let
  G⁺ = G with dummy vert s connected to all v ∈ V with weight 0
  D = BellmanFord (G⁺, s)
  w'(u,v) = w(u,v) + D[u] - D[v]      // always ≥ 0
  G' = (V, E, w')
  Dijkstra' u =
    let
      Δu = Dijkstra G' u
    in
      {(u,v) ↦ (d - D[u] + D[v]) : (v ↦ d) ∈ Δu}
    end
in
  ⋃_{u∈V} Dijkstra' u
end
```

Leftist Heap

rank Empty = 0

| Node (_, _, R) = 1 + rank R

Leftist: $\forall$ Node (L, _, R), rank L ≥ rank R

Heap: priority ordered vertically!

PQ impl	insert	delMin	meld	fromSeq
unsorted list	1	n	$m+n$	n
sorted list	n	1	$m+n$	$n \lg n$
balanced tree	$\lg n$	$\lg n$	$m \lg(1+\frac{n}{m})$	$n \lg n$
binary heap	$\lg n$	$\lg n$	$m+n$	n
leftist heap	$\lg n$	$\lg n$	$\lg m + \lg n$	n

→ always traverse down right spine for efficiency

```
mkLeftistNode (v, L, R) =
  if rank L < rank R then Node (1+rank L, R, L) else sym

meld Node(_, kA, LA, RA) Node(_, kB, LB, RB) =
  if kA < kB then mkLeftistNode (kA, LA, meld (RA, B)) else sym...
```

Lemma rank Q ≤ lg(|Q| + 1)

rank Q = r ⇒ |Q| ≥ $2^r - 1$ ← by induction

Graph Contraction

- Try contract constant fraction
- Usually root dominated

Greedy — factor 2 of optimal

Edge contract — $\frac{7n}{8}$ $\mathbb{E}$ size after contraction on cycle graph

$\forall e = \{u,v\} \in E$, flip coin, contract if head & all edges touching u, v tail

Star contract

flip coin on every vert, try pull tails to nearby head

$TH = \{\{u,v\} \in E \mid u \text{ tail} \wedge v \text{ head}\}$

$P_S = \bigcup_{\{u,v\} \in TH} \{u \mapsto v\}$ point to center

$V_C = V \setminus domain(P_S)$ centers

$P_C = \{u \mapsto u : u \in V_C\}$ center self loop

$\mathbb{E}$ satellites $\geq \frac{n}{4}$ for n non-isolated verts

Tree contract run multiple star contract...