Substitution

Recurrence

$$W(n) = 5W(\tfrac{n}{8}) + O(n^{\frac{2}{3}})$$
$$\leq 5W(\tfrac{n}{8}) + c_1 n^{\frac{2}{3}}$$

Guess

Leaf ... $O(n^{\log_8 5})$

Claim $W(n) \leq k_1 n^{\log_8 5} + k_2$

BC $W(1) \leq c_0 \leq k_1 + k_2$

IH $\forall_{1 \leq n' < n} \quad W(n') \leq k_1 n'^{\log_8 5} + k_2$

IS
$$W(n) \leq 5W(\tfrac{n}{8}) + c_1 n^{\frac{2}{3}}$$
$$\leq 5\left(k_1 (\tfrac{n}{8})^{\log_8 5} + k_2\right) + c_1 n^{\frac{2}{3}}$$
$$\vdots$$
$$= k_1 n^{\log_8 5} + 5k_2 + c_1 n^{\frac{2}{3}}$$
$$= k_1 n^{\log_8 5} + k_2 + 4k_2 + c_1 n^{\frac{2}{3}}$$

uh oh

Claim' $W(n) \leq k_1 n^{\log_8 5} + k_2 + k_3 n^{\frac{2}{3}}$

BC' ...

IH' $\forall_{1 \leq n' < n} \quad W(n') \leq k_1 n'^{\log_8 5} + k_2 + k_3 n'^{\frac{2}{3}}$

IS'
$$W(n) \leq 5W(\tfrac{n}{8}) + c_1 n^{\frac{2}{3}}$$
$$\leq 5\left(k_1 (\tfrac{n}{8})^{\log_8 5} + k_2 + k_3 (\tfrac{n}{8})^{\frac{2}{3}}\right) + c_1 n^{\frac{2}{3}}$$
$$\vdots$$
$$= k_1 n^{\log_8 5} + k_2 + k_3 n^{\frac{2}{3}} + \underbrace{4k_2 + c_1 n^{\frac{2}{3}} + \tfrac{1}{4} k_3 n^{\frac{2}{3}}}_{\text{now make this } \leq 0}$$

Define $k_2 = c_1$ $\quad k_3 = -20$ $\quad k_1 = 20$

Perf

T^* – sequential baseline

T_P – on P processors $\quad T_P \geq \frac{T_1}{P}$

Self-speedup $= \frac{T_P}{T_1} \leq P$

Overhead $= \frac{T_1}{T^*} \geq 1$

Efficient reduction := transformation takes no more asymptotic cost than solving transformed problem.

Impls

update (A (i, x)) changes A[i] to x

inject does update on a seq of (i, x)s. takes first one if duplicate index

```
reduce f I A =
  case |A| of
    0 => I |
    1 => A[0]
    n => let
        mid = ⌊n/2⌋
        (L,R) = (A[0 ... mid-1], A[mid ... n-1])
        (L',R') = (reduce L || reduce R)
      in
        f (L', R')
      end
```

```
scan f b S =
  case |S| of
    0 => (⟨⟩, b) |
    1 => (⟨b⟩, S[0]) |
    n => let
        cont = ⟨ f(S[2i], S[2i+1]) : 0 ≤ i < |S|/2 ⟩
        (R, t) = f b cont
      in
        (⟨ { R[i/2] if even i
           { f(R[i/2] + S[i-1])   : 0 ≤ i < |S|⟩, t)
      end
```

```
mcss A = let
  (b, v) = scan op+ 0 A
  prefixSums = append b ⟨v⟩
  (minPrefixes, _) = scan min ∞ prefixSum
  maxForEnds =
    ⟨ prefixSums[i] − minPrefix[i] : 0 ≤ i ≤ |A| ⟩
in
  reduce max -∞ maxForEnds
end
```

Log rules

$a^{k\log b} = b^{k\log a}$

$\log a^b = b\log a$

$a^{\log_a b} = b$

$\log_a b = \frac{\log_c b}{\log_c a}$

Recurrence banks

$T(n) =$

$T(\frac{n}{2}) + \log n \Rightarrow O(\log^2 n)$

$2T(n-1) + n \Rightarrow O(2^n)$

$3T(\frac{n}{2}) + n^{\frac{1}{2}} \Rightarrow O(n^{\log_2 3})$

$T = T(\sqrt[r]{n}) + \cdots$

$\overbrace{(\cdots(r^r)^r\cdots)^r}^{d} = n$

$r^{r^d} = n$

$r^d = \log_r n$

depth: $d = \log_r \log_r n$

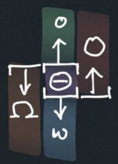
$T(n) = cT(\frac{n}{d}) + \cdots$

depth: $\log_d n$

leaves: $n^{\log_d c}$

Bound notations

$f(n)$



Rand Alg

		correct	time
Las Vegas	—	always	variable
Monte Carlo	—	maybe	constant

Associativity

Identity for max $-\infty$

min ∞

$O(\log\log n) \subset O(\log n) \subset O(\log^2 n)$

Nested parallel work-span model

greedy scheduling bound

$\max(\frac{W}{P}, S) \le T_P \le \boxed{\frac{W}{P} + S}$

Want $\frac{W}{P}$ dominate $\Rightarrow S < \frac{W}{P}$

$P < \underbrace{\left[\frac{W}{S}\right]}_{\text{parallelism}}$

Work efficient $:= W^{par}_{impl} \in O(W^{best}_{seq\ impl})$

Summations

$\sum_{i=0}^{n} i^a \in \Theta(n^{a+1})$

$\sum_{i=0}^{n} \frac{1}{i} \in \Theta(\log n)$

$\sum_{i=0}^{n} b^i \in \Theta(b^n)$

$\sum_{i=0}^{\log n} \frac{n}{2^i} \in \Theta(n)$ $\qquad \sum_{i=0}^{\log n} \frac{1}{2^i} < 2$

$\sum_{i=0}^{n} (\log^c i)(i^a)(b^i)$

$\in \Theta\left((n\log^c n)(n^a)(b^n)\right)$ $\quad b \ge 1$, $a, c \ge 0$

Wacky tree

$W(n) = W(\frac{n}{2}) + W(\frac{n}{3}) + 1$

Leaf Guess $L(n) = n^\beta$

$L(n) = L(\frac{n}{2}) + L(\frac{n}{3})$

$n^\beta = (\frac{n}{3})^\beta + (\frac{n}{3})^\beta$

Solve $\Rightarrow$ $\beta \approx 0.788$.

$O(n^{0.788})$

Prob

Random var $X : \Omega \to \mathbb{R}$

Expected var $\mathbb{E}[X]$ = weighted sum

Indep X, Y indep $\Leftrightarrow P[X=a, Y=b] = P[X=a]P[Y=b]$

Linearity $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Product expectation $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Union bound $P(A \cup B) \le P(A) + P(B)$

Conditional $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Markov inequality $X \ge 0 \Rightarrow P[X \ge a] \le \frac{\mathbb{E}[X]}{a}$

High Prob Bound $W(n) \in O(f(n))$ w.h.p. if $\exists$ constants c, n_0 s.t. $\forall n > n_0, \forall k, P[W(n) \le ckf(n)] \ge 1 - (\frac{1}{n})^k$

Sequences

$\langle \underset{\uparrow\ \text{map / tab}}{i} : 0 \le i < n \underset{\uparrow\ \text{filter}}{|} i \% 2 = 0 \rangle$ $\qquad A[1 \underset{\uparrow\ \text{inclusive}}{\cdots} 4]$ $\qquad A \underset{\uparrow\ \text{append}}{+\!\!+} B$

reduce $f\ b\ S$ usually $W(n) = 2W(\frac{n}{2}) + W_f(n)$

scan $f\ b\ S$ usually $W(n) = W(\frac{n}{2}) + W_{contract}(n) + W_{expand}(n)$

scan append $\langle \overbrace{\langle m\rangle, \langle m\rangle, \ldots, \langle m\rangle}^{n} \rangle$

	contract	recur	expand	solution
W	$2m \cdot \frac{n}{2} \in O(mn)$	$W(\frac{n}{2}, 2m)$	$O(n^2 m)$	$O(n^2 m)$
S	$O(1)$	$S(\frac{n}{2}, 2m)$	$O(1)$	$O(\lg n)$

Sorting $W = O(n\log n)$

$S = O(\log^2 n)$

scan op+ 0 $\langle 2\ 1\ \ 1\ 4\ \ 3\ 2\ \ 1\ 3 \rangle$

contract $\langle 3\ \ 5\ \ 5\ \ 4 \rangle$

recur $(\langle 0\ \ 3\ \ 8\ \ 13 \rangle, 17)$

expand $(\langle 0\ 2\ \ 3\ 4\ \ 8\ 11\ \ 13\ 14 \rangle, 17)$