$\boxed{\text{Lec 8}}$ Probability for randomised algorithms

Exam: bring yourself, 1 handwritten sheet, 4 function calculator

Motivation for randomised algorithms

- Can be faster
  - sometimes faster by constant factor
  - sometimes faster asymptotically
- Can be simpler
- Break symmetry — hopefully low probability to choose badly

- Unpredictable
  - Running time
  - Inconsistent
  - Don't know how long each fork takes when parallelised
- Need source of randomness
- Hard to analyse

Ex. Prime test randomised algorithm
Polynomial time, simple implementation

Las Vegas algorithm   random → always right answer

Monte Carlo alg   simulate → generate something close

Ex. Random Distance Run

2 giant dice
1st roll : how many laps for one
2nd roll : how many more to run

Define rand vars : D_1 = value of 1st die
D_2 = value of 2nd die

Expected values... $E[D_1] = 3.5$
$E[D_2] = 3.5$

Multiplication works if independent

What about expected sum of 2 dice $E[D_1 + D_2] = 7$
... expected product of 2 dice ... $E[D_1 D_2] = \cdots 12.25$
... expected max $E(\max(D_1, D_2)) = 4\frac{17}{36}$ ← Not clearly related to max of expectation.

Probability

<u>Sample space</u> Ω

<u>Prob measure</u> $P : \mathcal{P}(\Omega) \to \mathbb{R}$ with:

1. $\forall A, \quad 0 \leq P(A) \leq 1$
2. $\forall A, B, \quad A \cap B = \emptyset \Rightarrow P(A) + P(B) = P(A \cup B)$
3. $P(\Omega) = 1$

<u>Random variable</u> ... neither random nor variable

Deterministic function $X : \Omega \to \mathbb{R}$

<u>Expected value</u> of X $\quad E[X] = \sum_{\omega \in \Omega} \underbrace{P(\omega)}_{\text{prob of that outcome}} \cdot \overbrace{X(\omega)}^{\text{value of outcome}}$

<u>Independent</u> $\quad X, Y$ indep if

$$P[X = a, Y = b] = P[X = a]\,P[Y = b] \quad \forall a, b$$

<u>Linearity of expectation</u> $\quad E[X + Y] = E[X] + E[Y] \leftarrow$ (always true)

<u>Expectation of product</u> $\quad E[X \cdot Y] = E[X] \cdot E[Y] \leftarrow$ (assuming independent)

<u>Union bound</u> $\quad P(A) + P(B) \geq P(A \cup B)$

<u>Conditional prob</u> $\quad P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Entangled dice

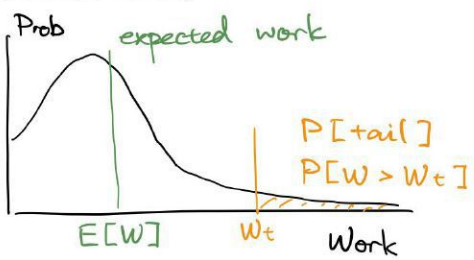
Suppose 2nd die must be same as first die

Expected sum of dice $\longrightarrow$ 7

Expected product $\longrightarrow$ $15\frac{1}{6}$

Alg analysis with prob

<u>Tail bound</u>



Markov's inequality   tool for bounding tail

If $X \geq 0$ then $P[X > a] \leq \frac{E[X]}{a}$ $\forall a$ ← threshold

Quicksort

pick random pivot → partition → recur → append
Unlucky case : picking bad pivot .

Goal : analyse work & span of rand. alg.

W_1, S_1 W_1, S_2

$W = W_1 + W_2$ ← okay to bound
$S = \max(S_1, S_2)$ ← hard to bound

High probability bound

Say $W(n) \in O(f(n))$ with high probability (w.h.p) if
$W(n) \in O(k \cdot f(n))$ with probability $> 1 - \left(\frac{1}{n}\right)^k$

how much worse ⟷ how often does it violate bound
We can define these differently

Intuitively, $k \uparrow$ $1 - \left(\frac{1}{n}\right)^k \uparrow$
so the higher the violation the less often we are allowed to violate the bound

Consider max of n spans