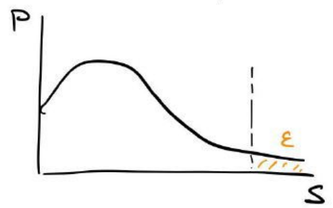
Lec 9 Probability Bound Analysis

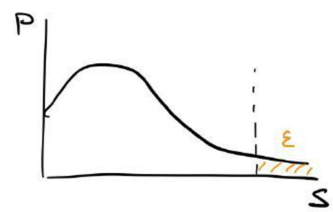
High prob bound

<u>Redef</u> Say $W(n) \in O(f(n))$ w.h.p. if $\exists$ constants c, n_0 s.t.
$\forall n > n_0$, $\forall k$ $W(n) \leq c k f(n)$ with probability $\geq 1 - (\frac{1}{n})^k$

Max of spans

Consider n spans





Suppose there's ε prob. that a single span is bad

Prob that some of them bad is by union bound $\leq n\varepsilon$

$P[\text{Some bad}] = P[1^{st} \text{ being bad} \cup \cdots \cup n^{th} \text{ being bad}]$

Ex. Suppose each piece has $O(\lg n)$ w.h.p.

$P[\text{indiv good}] = 1 - (\frac{1}{n})^k$

$P[\text{indiv bad}] = (\frac{1}{n})^k$

$$P[\text{some bad}] \leq n(\frac{1}{n})^k = (\frac{1}{n})^{k-1}$$
$$= (\frac{1}{n})^{k'} \quad \forall k'$$

$\Rightarrow$ Overall span is $O(\lg n)$ with prob $\geq 1 - (\frac{1}{n})^{k'}$
so w.h.p. composed to $O(\lg n)$

Ex. toy alg. for skittles game

Game: jar start with n skittles
flip coin, if head eat [half of remaining]
else noop

Question: how many rounds before run out of skittles

... worse case ∞ ?

Define random var X_d := number of skittles at start of round d.

$$X_0 = n$$

$$\mathbb{E}[X_{d+1}] = \frac{1}{2}\,\mathbb{E}[X_d] + \frac{1}{2}\left\lfloor \frac{\mathbb{E}[X_d]}{2} \right\rfloor$$

$$\leq \frac{3}{4}\,\mathbb{E}[X_d]$$

$$\Rightarrow \mathbb{E}[X_d] \leq n\left(\frac{3}{4}\right)^d$$ by induction

Claim: num rounds $\leq 10 \lg n$ with prob $1 - \left(\frac{1}{n}\right)^{3.15}$

Proof.

$$\mathbb{E}[X_{10\lg n}] \leq n\left(\frac{3}{4}\right)^{10\lg n}$$

$$= n \cdot n^{10 \lg \frac{3}{4}}$$

$$\approx n \cdot n^{-4.15}$$

$$= \frac{1}{n^{3.15}}$$

By Markov's inequality $P[X_{10\lg n} \geq 1] \leq \frac{\mathbb{E}[X_{10\lg n}]}{1} = \frac{1}{n^{3.15}}$

$$\Rightarrow P[X_{10\lg n} < 1] = P[X_{10\lg n} = 0]$$

$$> 1 - P[X_{10\lg n} \geq 1]$$

$$= 1 - \frac{1}{n^{3.15}}$$

Lemma: num of rounds $\leq \frac{-(k+1)}{\lg(\frac{3}{4})} \lg n$ with prob $\geq 1 - (\frac{1}{n})^k$

let $c = \frac{-(k+1)}{\lg(\frac{3}{4})}$. $\mathbb{E}[X_{c\lg n}] \leq n\left(\frac{3}{4}\right)^{c\lg n}$

$$= n \cdot n^{c \lg \frac{3}{4}}$$

$$= n \cdot n^{\left(\frac{-(k+1)}{\lg(\frac{3}{4})}\right)\lg\frac{3}{4}}$$

$$= n \cdot n^{-(k+1)}$$

$$= \frac{n}{n^{k+1}}$$

$$= \left(\frac{1}{n}\right)^k$$

markov stuff ... $\in O(\lg n)$

Analysing random select

An order statistics problem

Given seq A and rank k, return k^{th} smallest elem of A

→ One can simply sort, but not efficient enough

└ $W = O(n \lg n)$, $S = O(\lg^2 n)$

→ Goal: $W = O(n)$, $S = (\lg^2 n)$ w.h.p.

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rselect A k = let
  p = uniformly randomly selected elem
  (L,R) = ⟨x∈A : x<p⟩ || ⟨x∈A : x>p⟩
in
  if k < |L| then select L k
  elif k = |L| then p
  else rselect R (k - |L| - 1)
```

Randomised select by contraction

Partition by pivot, then the cases...

① p is the k^{th}

② L longer than k ⇒ recurse on L

③ L shorter than k ⇒ recurse on R

Intuition for analysis

→ 50% of time picking p between Q1 and Q3

in that case we eliminate 25% of elems