Lec 10 Random Algorithm II – Order Stats Problem Analysis

Recall: skittle game, search for k-th rank in list

Randomised Select Analysis

```
rselect A k = let
  p = uniformly randomly selected elem
  (L,R) = <x∈A: x<p> || <x∈A: x>p>
in
  if k < |L| then select L k
  elif k = |L| then p
  else rselect R (k-|L|-1)
```

Lucky: pick pivot close to median and eliminate $\frac{1}{2}$
Unlucky: pick close to min / max and eliminate 1
Midluck: pick sth between and eliminate $\frac{1}{4}$

Input size unknown...

at level d 0 1 2 3 .... n-2 n-1 n

possible size decreases for given input size

at level $d+1$ 0 1 2 3 --- n-2 n-1 n

Let Y_d be RV for input len at level d $(Y_0 = n)$
Z_d be RV for rank of pivot chosen at level d.

$$\mathbb{E}[Y_{d+1}] = \sum_y \sum_z \underbrace{P[Y_d = y, Z_d = z]}_{\text{prob of having input size } y \text{ and picking rank } z \text{ at previous level. Corresponds to each edge!}} \underbrace{f(y,z)}_{\text{len of remaining input?}}$$

$$= \sum_y \sum_z P[Y_d = y] \underline{P[Z_d = z \mid Y_d = y]} f(y,z)$$

$$= \sum_y \sum_z P[Y_d = y] \frac{1}{y} f(y,z)$$

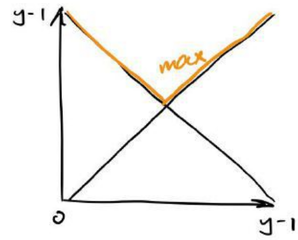
$$= \sum_y \left[P[Y_d = y] \sum_z \frac{1}{y} \underline{f(y,z)} \right]$$

$f(y,z)$ needs to return remaining input size

z	possible $f(y,z)$
0	$0, \; y-1$
1	$0, 1, y-2$
2	$0, 2, y-3$
$\vdots$	
z	$0, z, y-z-1$
$\vdots$	
$y-2$	$0, 1, y-2$
$y-1$	$0, y-1$

Worse case...

$$\sum_z f(y,z)$$
$$= \sum_{z=0}^{y-1} \max(0, z, y-z-1)$$
$$= 2\sum_{z=y/2}^{y-1} z$$
$$\vdots$$
$$\leq \frac{3}{4} y^2$$



$$\leq \sum_y \left[P[Y_d = y] \, \frac{1}{y} \, \frac{3}{4} y^2 \right]$$

$$= \frac{3}{4} \sum_y P[Y_d = y] \, y$$

$$= \frac{3}{4} \mathbb{E}[Y_d]$$

So $\mathbb{E}[Y_d] \leq n \left(\frac{3}{4}\right)^d$

Expected work $\mathbb{E}[W] = \mathbb{E}[W_0 + \cdots + W_n]$

$$= \sum_{d=0}^{n} \mathbb{E}[W_d]$$

$$= \sum_{d=0}^{n} O\left(n \left(\tfrac{3}{4}\right)^d\right)$$

$$\in O(n)$$

Expected span $\mathbb{E}[S]$

$\mathbb{E}[\#\text{ of levels}] \in O(\lg n)$ w.h.p. (same as skittles game)

$\Rightarrow \mathbb{E}[S] \in O(\lg^2 n)$

Quicksort

```
qsort A = if |A| ≤ 1 then A else let
   p = uniformly selected pivot
   L,R = partition in parallel
   L',R' = qsort L || qsort R
in L' ++ ⟨p⟩ ++ R' end
```

Analysis by counting the number of comparisons

Define RVs $X_{i,j} = \begin{cases} 0 & \text{if keys ranked } i,j \text{ never compared} \\ 1 & \text{if } \ldots \text{ are compared} \end{cases}$

↑ Indicator RV

Observe : the pivot gets compared to everything
things only get compared if they get picked as pivot
and they they don't get compared in recursive calls

if $x < y < z$ and y is pivot, x and z never get compared

WLOG $i < j$

$\mathbb{E}[X_{i,j}] = P[X_{i,j} = 1] = \frac{1}{j-i+1}(2!)$

order for choosing i,j

chance for picking i, j in

$\mathbb{E}[W] = O(\mathbb{E}[\#\text{ of comparisons}])$

$= O(\sum_{i<j} \mathbb{E}[X_{i,j}])$

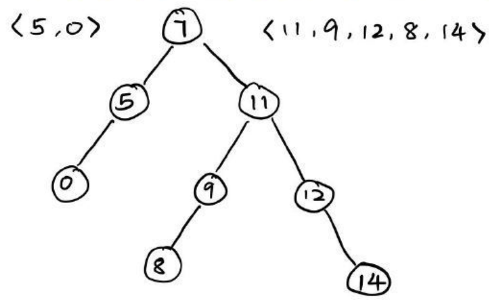
$\leq 2 \sum_{i=0}^{n} H_i$ ← harmonic number

$\in O(n \log n)$

$\mathbb{E}(S)$ analysis by pivot tree

recursion tree showing the pivot chosen at each node

$\langle 7, 5, 11, 0, 9, 12, 8, 14 \rangle$, always picking first



from randomised select, we found the length of one path is $O(\lg n)$ w.h.p.

$P[\text{one path} > k_1 \lg n] \leq \frac{1}{n^{k_1}}$ for all constant k_1

WTS $P[\exists \text{ path} > k_2 \lg n] < \frac{1}{n^{k_2}}$ for all constants k_2

But there are $< n$ paths. By union bound:

$P[\exists \text{ path} \geq k_2 \lg n] \leq n \cdot \frac{1}{n^{k_1}}$ for all k_1.

$\leq \frac{1}{n^{k_2}}$ as long as we choose $k_1 = k_2 + 1$