Lec 15 — Graph Search / Graph Traversal

Recall: $G = (V, E)$
$N_G(u) = \{\text{nbors of } u\}$
$d_G(u) = \text{degree of } u$
$\delta_G(u,v) = \text{distance from } u \text{ to } v$

Generic graph search

<u>Def</u> Graph search / traversal is when we systematically examine nodes in a graph starting from some vert v.

<u>Def</u> $R_G(s) = \{ v \in V \mid v \text{ reachable from } s \}$

<u>Generic traversal</u>

Keep track of:

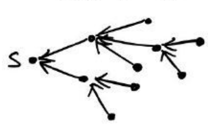
- visited $X = \{\text{set of visited}\} \subseteq V$
- frontier $F = \{\text{next to some visited node but not visited}\} \subseteq V \setminus X$

Alg:

```
search G s :
  X = {}
  F = {s}
  while |F| > 0 :
    U = some non-empty subset of F
    visit everything in U
    X = X ∪ U
    F = N⁺_G(X) \ X
  return X
```

<u>Thm</u> search G s returns $R_G(s)$

<u>Def</u> graph search tree is a graph built by
for $v \in R_G(s)$:
create edge from v to the vertex that added it to v (or some vert if race condition)



Cost Analysis

$X = X \cup U$ ← union
$F = N^+_G(X) \setminus X$ ← finding nbors & set diff

<u>Claim</u> cost is dominated by $N^+_G(X)$
(assume for now)

$$N^+_G(X) = \bigcup_{v \in X} N^+_G(v)$$

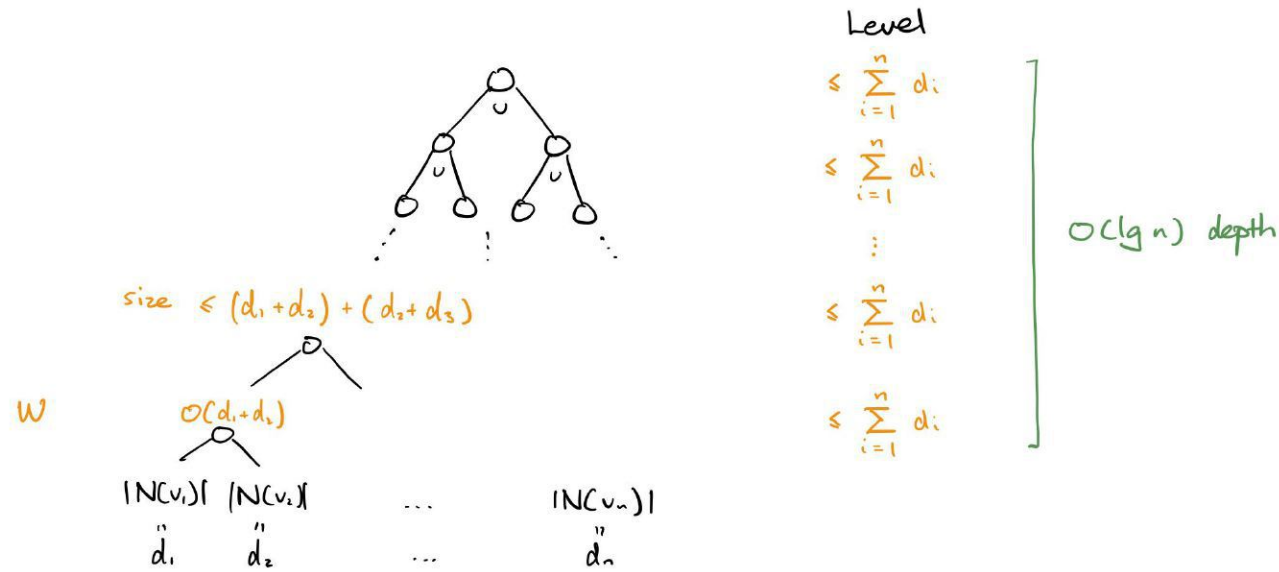
$$= \text{reduce union } \emptyset \ \langle N^+_G(v) : v \in X \rangle$$

Recall Assume $a = |A| \le b = |B|$

(exercise)

$$W_{union}(a,b) = O(a \lg(\tfrac{b}{a}+1)) \le O(a+b)$$
$$S_{union}(a,b) = O(\lg(a+b))$$

Reduce union recursion tree



But $\sum_{i=1}^{n} d_i \le O(m)$ so total work $O(m \lg n)$

⋮

Span: at level $\le O(\lg n)$
overall $O(\lg^2 n)$ ← (is tight bound)

Parallel BFS

When $U = F$

BFS G s = let
  loop (X, F, i) =
    if $|F| = 0$ then (X, i)
    else Δ let
      $X' = X \cup F$
      $F' = \underbrace{N_G^+(F) \setminus X'}$ — for BFS this $\equiv N_G^+(X') \setminus X'$
      actually lower cost to get all nbors
    in
      loop $(X', F', i+1)$
    end
  in
    loop $(\{\}, \{s\}, 0)$
  end

Claim: at Δ,
$X_i = \{ v \in V, \ \delta(s,v) < i \}$
$F_i = \{ v \in V, \ \delta(s,v) = i \}$

$\cdot i := \cdot$ when loop called with counter i

Proof It feels right $\square$ (no)

Proof by induction

BC $i = 0$, $X_0 = \emptyset$. ✓
$F_0 = \{s\}$. ✓

IS Assume for i, WTS for $i+1$

$$
\begin{aligned}
X_{i+1} &= X_i \cup F_i \\
&= \{ v \in V, \ \delta(s,v) \le i \} && \text{(IH)} \\
&= \{ v \in V, \ \delta(s,v) < i+1 \} && ✓
\end{aligned}
$$

$$
\begin{aligned}
F_{i+1} &= N_G^+(F_i) \setminus X_{i+1} \\
&= \{ v \in V, \ \delta(s,v) = i+1 \} && ✓
\end{aligned}
$$

BFS on line graph

$\bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet$

iterations $O(n)$, $O(\lg n)$ at each iter

$W = O(n \lg n)$
$S = O(n \lg n)$

BFS cost

let $\|F\| = \sum_{x \in F} (d^+(x) + 1)$ assume tree set

for iter i

	W	S
- $X_i \cup F_i$	$O(\lvert F_i \rvert \lg n)$	$O(\lg n)$
- $N_G^+(F_i)$	$O(\|F_i\| \lg n)$ same analysis as above	$O(\lg^2 n)$
- $\setminus X_{i+1}$	$O(\lvert F_i \rvert \lg n)$	$O(\lg n)$