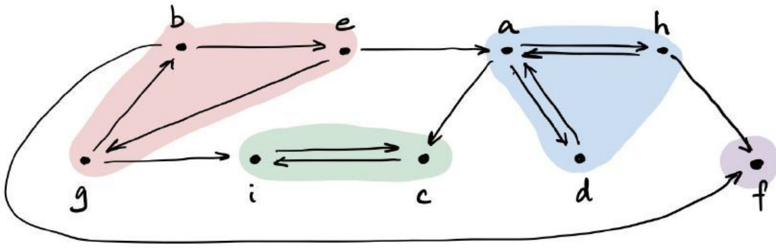
Lec 17 Kosaraju's Alg & Shortest path

Strongly connected component (SCC)

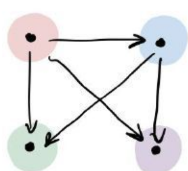
__Def__ A subset of verts $S \subseteq V$ is strongly connected (SC) if
$\forall v \in S, \forall u \in S$, u reachable from v

__Def__ If $S \subseteq V$ is SC and is maximal, it's a strongly connected component

__Ex.__

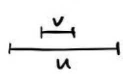


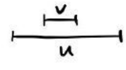
__Claim__ Contracting the SSCs gets you a DAG ← else one of those SSC not maximal.
↳ turn SSCs into verts and add edge by reachability btwn components

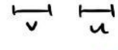


__Def__ The SCCs problem: find the SCCs in graph and return them in topological order

__Lemma__ For directed graph G, if $u \in V$ is first-visited vert in its SCC, all verts v reachable from u finish before u in a DFS

__C1__ v, u in same SCC $\Rightarrow$ 

__C2__ v, u in diff SCC $\wedge$ u visited before v $\Rightarrow$ 

__C3__ v, u in diff SCC $\wedge$ u visited after v $\Rightarrow$ 

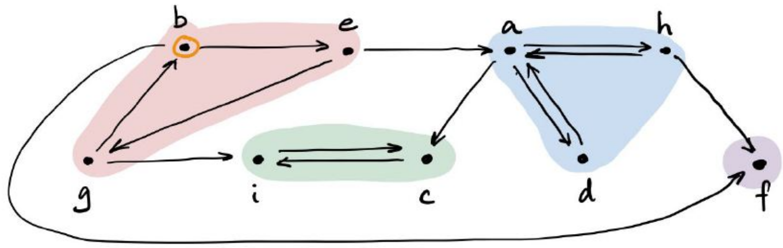
... otherwise u, v in same SCC so we can't go back

Kosaraju's Algorithm

```
SCC G = (V, E) =
  let
    F = reverseFinishTime G
    G^T = transpose G                 ( reverse the edges )
                      visited verts   SCC so far
    accumSCCs ( (X, L), u ) =
      let  overall visited
        (X', A) = reach G^T X u
          newly visited   search for all reachable from X
        ▷ If A = ∅ ⇒ we saw new SCC
      in
        if A = ∅ then (X, L) else
        (X', L ++ ⟨A⟩)
  in
    iterate accumSCC (∅, ⟨⟩) F
  end
```

Cost 2× DFS, so actually linear time

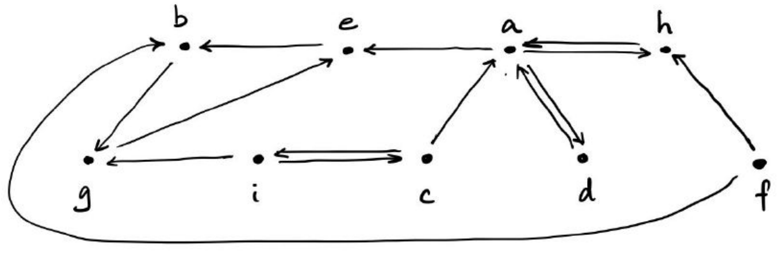
Trace



$F \Rightarrow \langle \underline{b}\ e\ g\ a\ h\ f\ d\ c\ i \rangle$

latest finish

$G^T \Rightarrow$



			A	L
reach G^T $\varnothing$	b	$\Rightarrow$	$\{b, e, g\}$	$\langle \{b, e, g\} \rangle$
reach G^T $\{b, e, g\}$	e	$\Rightarrow$	$\varnothing$	"
reach G^T $\{b, e, g\}$	g	$\Rightarrow$	$\varnothing$	"
reach G^T $\{b, e, g\}$	a	$\Rightarrow$	$\{a, d, h\}$	$\langle \{b, e, g\}, \{a, d, h\} \rangle$
" $\{b, e, g, a, d, h\}$	h	$\Rightarrow$	$\varnothing$	"
" "	f	$\Rightarrow$	$\{f\}$	$\langle \{b, e, g\}, \{a, d, h\}, \{f\} \rangle$
" $\{b, e, g, a, d, h, f\}$	d	$\Rightarrow$	$\varnothing$	"
" "	c	$\Rightarrow$	$\{c, i\}$	$\langle \{b, e, g\}, \{a, d, h\}, \{f\}, \{c, i\} \rangle$

{ no more to add

Correctness

Observe:
When first reach each SCC U_i via vert u_i in rev finish order:

1. SCCs left of U_i already completely visited
2. reach G^T u_i will not visit any SCCs to right of U_i
3. reach G^T u_i will visit all verts in U_i

where left / right identified by first appearance of vert in SCC in F

$$\langle \underset{\substack{\uparrow \\ u_1}}{b}\ e\ g\ \underset{\substack{\uparrow \\ u_2}}{a}\ h\ \underset{\substack{\uparrow \\ u_3 \\ \triangle}}{f}\ d\ \underset{\substack{\uparrow \\ u_4}}{c}\ i \rangle$$

Notice 1-3 $\Rightarrow$ reach G^T u_i visit exactly U_i

If $\triangle$ is current, all of $\leftarrow$ unreachable from $\triangle$ in G
otherwise F is not rev finish time order
$\Rightarrow$ $\triangle$ unreachable from any of $\leftarrow$ in G^T

Shortest path problem

Def weighted graph has some weight for each edge

$G = (V, E, w)$ $\quad w : E \to \mathbb{R}$

one representation G : (V, (V, $\mathbb{R}$) table) table

$\delta(u, v) :=$ shortest path with min edge weights from u to v

Def Single-pair shortest path problem (SPSP)

Given u, v, find $\delta(u,v)$

Def Single-source -- (SSSP)

Given u, find $\delta(u,v)$ $\forall v$ reachable from u

Def All-pairs --

$\forall u, \forall v$, find $\delta(u,v)$

Priority-first search (PFS) aka best-first search

Decide where to search by order returned by some priority func

viz. pick $v \in F$ by highest priority

Ex. beam search, A^*, dijkstra

sth in F

$p(v)$

priority

Dijkstra's property :

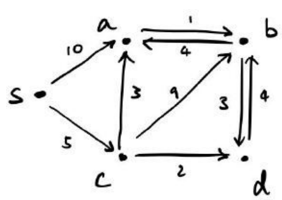
if $\nexists$ neg edge weights in G, let $p(v) = \min_{u \in X} (\delta(s,u) + w(u,v))$

then $v \in V \setminus X$ with smallest $p(v)$ has $\delta(s,v) = p(v)$

Dijkstra's algorithm :

use the above p as priority in PFS (record $p(v)$ for each vert v when we visit v)

Ex.



s	a	b	c	d
0	∞	∞	∞	∞
	10	∞	5	∞
	8	14		7
	8	11		
		9		

$\delta(s, s) = 0$

$\quad s \;\; c = 5$

$\quad s \;\; d = 7$