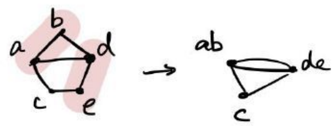
# Lec 20 Star Contraction & Connectivity

# Graph contraction

parallelism, polylog span, root dominated work
contract to get constant fraction smaller




<u>Defs</u> Graph partition := subgraph $H = (V', E')$ with $V' \subseteq V$ and $E' = \{\{u,v\} \in E \mid u, v \in V'\}$ viz. cut out verts and keep reasonable edges

Given partitions $H_1, \dots, H_{k-1}$, $\{u,v\} \in E$ is
- internal edge if $u, v \in V_i$
- cut edge if $u \in V_i$, $v \in V_j$, $i \neq j$

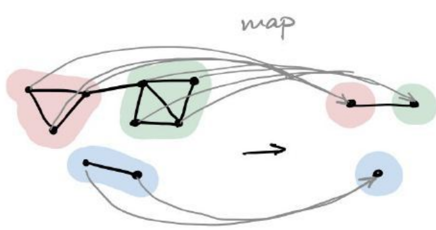
Quotient graph is contracted, smaller graph

Supervert is vert in quotient that verts in orig graph "merged" to

<u>Repr</u> 1. Label for each part
2. Map from vert to their part label

<u>General Contraction</u>

<u>BC</u> Small graph ⇒ compute result




<u>IC</u> <u>Contract</u> Make quotient
- Partition
- Turn part into vert
- Drop internal edges
- Point cut edges elsewhere (maybe remove dups)

<u>Recur</u> Solve on contracted graph

<u>Expand</u> Get result for bigger graph

# Edge contraction

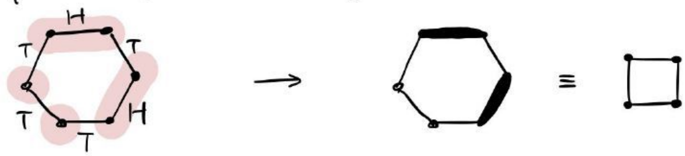
When each part is a vert or one edge.

First we need to find matching

→ Greedy : for $e \in E$, keep adding $e$ to $M$ if possible ↙ sequential, always within factor of 2 of optimal
→ Random : parallel assignment, local decisions

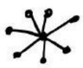
<u>Coin flip</u>

Flip coin for each edge, contract head s.t. no neighbouring edge is head




This gives constant fraction on some graph but not others

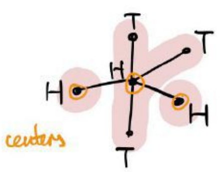
→ Cycle graph  each edge $\frac{1}{8}$ prob contracted, so $\mathbb{E}$ contract $\frac{m}{8}$

→ Star graph  then we only contract max 1 edge.

# Star Contraction

Each part looks like a star with center & satellites

→ Sequential : pick center, add all nbors as satellites, remove, repeat
→ Random : flip coin on each vert, turn H into centers, for each T try to contract into neighbouring H, then turn T that failed to merge into center




```
starPart (G = (V,E)) =
  let
    TH = { (u,v) ∈ E | u tail ∧ v head }
    Ps = ∪_(u,v)∈TH { u ↦ v }          ← point sats to centers
    Vc = V \ domain(Ps)                ← center verts
    Pc = { u ↦ u : u ∈ Vc }            ← center contract to center
  in
    (Vc, Ps ∪ Pc)
  end
```

<u>Cost</u> $W = O(n+m)$
$S = O(\lg n)$ ] with array seq

<u>Fact</u> for $G$ with $n$ non-isolated verts, $\mathbb{E}$ satellites $\geq \frac{n}{4}$

Contraction alg

```
starContract base expand (G=(V,E)) =
  if |E|=0 then base V
  else let
    (V', P) = starPart (V, E)
    E' = {(P[u], P[v]) : (u,v)∈E | P[u] ≠ P[v]}   remove self edge
    R = starContract base expand (V', E')
  in
    expand (V, E, V', P, R)
  end
```

Cost ( star contract until $|E| = 0$ )

Assume : $W_{base} = O(|V|)$ $\quad S_{base} = O(1)$

$W_{expand} = O(|V|+|E|)$ $\quad S_{expand} = O(\lg(|V|+|E|))$

$W = O((m+n)\lg n)$ $\qquad S = O(\lg^2 n)$

# Application : Graph Connectivity

Prob  Given undirected G, find all CCs by specifying them as vert set

→ Could do BFS or DFS, but slow

Contraction alg

```
connectedComponents (G = (V,E)) =
  if |E| = 0 then (V, {v ↦ v : v∈V})
  else let
    (V', P) = starPart (V, E)
    E' = {(P[u], P[v]) : (u,v)∈E | P[u] ≠ P[v]}
    (V'', C) = connectedComponents (V', E')
  in
    (V'', {u ↦ C[v] : (u↦v) ∈ P})
  end
```