L.T.

$T(\sum c_i v_i) = \sum c_i T(v_i)$

Inverse

$A^{-1}_{n\times n}$ s.t. $AA^{-1} = I_n = A^{-1}A$

Inverse assuming $\exists A^{-1}, B^{-1}$

$(cA)^{-1} = \frac{1}{c}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

$(A^n)^{-1} = (A^{-1})^n$

FTIM

TFAE

- $\exists (A^T)^{-1}$
- $\exists A^{-1}$
- rref $A = I_n$
- $Ax = b$ unique sol
- $Ax = \vec{0} \Rightarrow x = \vec{0}$
- $A = E_1 \cdots E_N$
- $\det A \neq 0$
- A non-singular
- 0 is not an e-val
- rank $A = n$ & nullity $A = 0$
- Cols & rows of A basis for $\mathbb{R}^n$

Det

$\det AB = \det A \det B$

$\det A = \det A^T$

Same as cols:

- Swap row — $\times -1$
- Mul row by c — $\times c$
- $r_1 \mapsto r_1 + cr_2$ — $\times 1$
- Linear func of rows

If $\exists$ row of 0, $\det = 0$

If upper triangular $\prod$ [diag]

$\det A = \det A^T$ $\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix}$

Proj

$\text{proj}_u v = \frac{u \cdot v}{u \cdot u} u$

$\text{proj}_{\text{col}A} v = A(A^TA)^{-1}A^T v$

$\text{proj}_W v = \sum$ proj onto orthal basis of W

Eigen

$Av = \lambda v \quad (v \neq 0)$

$\det(A - \lambda I) = 0$

$\begin{bmatrix} \cdot-\lambda & \cdots & \\ & \cdot-\lambda & \\ \vdots & & \ddots \end{bmatrix} v = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$

$E_\lambda = \text{nul}(A - \lambda I)$

matrix	eval	evec
A	λ	
A^n	λ^n	v
A^{-1}	λ^{-1}	
A^{-n}	λ^{-n}	

$A_{n\times n}$ with $\lambda_1 \cdots \lambda_n$, $v_1 \cdots v_n$

Then $A^m(\sum c_i v_i) = \sum c_i \lambda_i^m v_i$

Matrix Mul Reps

$\begin{bmatrix} \equiv \end{bmatrix} \begin{bmatrix} |||| \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & (\cdot\cdot) & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$ row-col

$A \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Ac_1 & \cdots & Ac_n \\ | & & | \end{bmatrix}$ matrix-col

$\begin{bmatrix} - r_1^T - \\ \vdots \\ - r_n^T - \end{bmatrix} B = \begin{bmatrix} - r_1^T B - \\ \vdots \\ - r_n^T B - \end{bmatrix}$ row-matrix

$\begin{bmatrix} |||| \end{bmatrix} \begin{bmatrix} \equiv \end{bmatrix} = \square + \cdots + \square + \cdots + \square$ col-row

IP

$\langle u, v \rangle = \langle v, u \rangle$

$\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

$\langle cu, v \rangle = c\langle u, v \rangle$

$\begin{cases} \langle u, u \rangle \geq 0 \\ \langle u, u \rangle = 0 \Leftrightarrow u = \vec{0} \end{cases}$

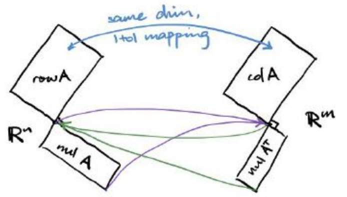
$\|u\| = \sqrt{\langle u, u \rangle} \quad \cos\theta = \frac{\langle u, v \rangle}{\|u\|\|v\|}$

Row Col Nul Perp Fun

A^TA invertible $\Leftrightarrow$ cols of A lin indep.

For $A_{m\times n}$:

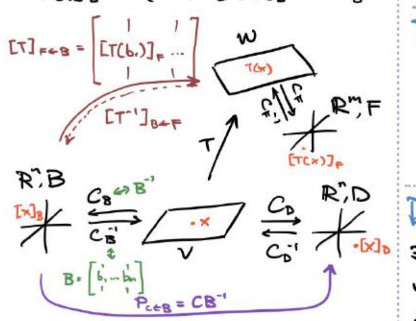
$(\text{row}A)^\perp = \text{nul}A \qquad \text{nul}A^T = (\text{col}A)^\perp$



Change of Basis

$B = \{b_{1 \ldots n}\}$, $D = \{d_{1 \ldots n}\}$

$[b_i]_B = e_i \dashrightarrow B[x]_B = [x]_d$



$P_{C\leftarrow B} = \begin{bmatrix} | & & | \\ [b_1]_C & \cdots & [b_n]_C \\ | & & | \end{bmatrix}$

Similar Transformation

$\exists P$ s.t. $P^{-1}AP = B \Rightarrow A \sim B$

$\sim$ is eq. rel.

If $A \sim B$:

- $\det A = \det B$
- $\exists A^{-1} \Leftrightarrow \exists B^{-1}$
- rank A = rank B
- A, B have same $\det(M - \lambda I)$
- same e-vecs

Orth. Matrix

:= orthonormal cols

$Q_{m\times n} \rightarrow Q^TQ = I_n$, $\|Qx\| = \|x\|$

$Q_{n\times n} \rightarrow Q^{-1} = Q^T$, $(Q\vec{x})\cdot(Q\vec{y}) = \vec{x}\cdot\vec{y}$

Q, R orth matrix $\Rightarrow$

- $Q^T = Q^{-1}$ orth
- $\det Q = \pm 1 \quad \lambda = \pm 1$
- QR orth

* Permutation matrix $\begin{bmatrix} \text{shuffle} \\ e_1 \cdots e_n \end{bmatrix}$

Transpose

$(AB)^T = B^TA^T$

$(kA)^T = k(A^T)$

$(A^n)^T = (A^T)^n$

$[A \mid I_n] \rightsquigarrow [I_n \mid A^{-1}]$

$[C \mid B] \rightsquigarrow [I_n \mid C^{-1}B]$

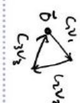
Least Square

Solve $A^Tb = A^TA x^*$

If cols of A lin indep

$A^+ = (A^TA^{-1})A^T$

Lin dep.



non trivil combo goes back $\vec{0}$

$\text{nul}A = \{x \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$

$\text{col}A = \{Ax \mid x \in \mathbb{R}^n\}$

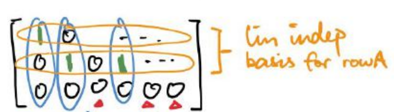
$W^\perp = \{v \in V \mid \forall w \in W, \langle v, w \rangle = 0\}$

kernel $T = \{v \in V \mid T(v) = \vec{0}\}$

Subspace (of matrix)

- Has $\vec{0}$
- Closed under $\odot, \oplus$

- Row op doesn't change row A nor dep rel between A's cols.
- nul A = nul R, row A = row R



Basis for col R.

Same place in A $\rightarrow$ colA

rank = 3 $\xrightarrow[+]{6=n}$ nullity = 3

- Solve $Rx = 0$ to get $B_{\text{nul}A}$

$1 \leq$ geo mul.ty $\leq$ alg mul.ty

dim(E_λ) — times root appear

- rank A = rank A^T

Basis

- Lin indep spanning set
- Gives unique way to write $v \in V$

Thm $B = B^W \cup B^{W^\perp}$ is orth basis for $\mathbb{R}^n$ (orth basis)

Diagonalisability

$P_{B\leftarrow d}[T]_{d\leftarrow d}P_{d\leftarrow B} = D$

T wrt e-basis looks like diagonal

$\exists D, P, P^{-1}, \quad P^{-1}AP = D$

viz. similar to some diagonal matrix

$A_{n\times n}$ diagonizable $\Leftrightarrow$ A has n lin indep e-vecs.

Then $P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

$B = \bigcup B_i \qquad \forall \lambda_i$, geo mult. = alg multi

Basis for E_{λ_i}

Orthogonally diagonalisable if P is orth. matrix Q.

$Q^{-1}AQ = D = Q^TAQ$

symmetric $\Leftrightarrow$ Orthogonally diagonalisable

Std form: $a_{11}x_1 + \cdots + = b_1$, $\vdots$

Normal form: $[\ ]\cdot v = b_1, [\ ]\cdot v = b_2 \cdots$