Lec 4

* Theorem: Algebraic properties of $\mathbb{R}^n$.

If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$ and $\vec{0} \in \mathbb{R}^n$ is a zero vector.
Then:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
- $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
- $c(d\vec{u}) = (cd)\vec{u}$

Different addition! Sometimes $\oplus$ to indicate vector addition

Vector Space

$:= (V, \oplus, \odot)$

V := set of vectors that satisfies these axioms:

$\forall \vec{u}, \vec{v}, \vec{w} \in V$, $\forall c, d \in \mathbb{R}$,

1. V is closed under addition i.e. $\vec{u} \oplus \vec{v} \in V$
2. $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$
3. $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$
4. $\exists \vec{0} \in V$, $\vec{u} \oplus \vec{0} \in V$
5. $\exists -\vec{u} \in V$, $\vec{u} \oplus (-\vec{u}) = \vec{0}$
6. $c \odot \vec{u} \in V$ ← closed under scalar mult
7. $c \odot (\vec{u} \oplus \vec{v}) = c \odot \vec{u} + c \odot \vec{v}$
8. $(c + d) \odot \vec{u} = c \odot \vec{u} + d \odot \vec{u}$
9. $c \odot (d \odot \vec{u}) = (cd \odot \vec{u})$
10. $1 \odot \vec{u} = \vec{u}$

$\mathbb{R}^n$ look like this, but there exist other vector spaces that satisfy these props.

$\oplus$:= addition rule

$\odot$:= scalar mult rule

Ex. Consider $P = \{$ polynomials in x with real coefficients $\}$

Let $p = a_0 + a_1 x^1 + \cdots + a_n x^n$
$q = b_0 + b_1 x^1 + \cdots + b_m x^m$

Define $\oplus$ by $p \oplus q = (a_0 + b_0) + (a_1 + b_1) x^1 + \cdots$ [normal way]

$\odot$ by $c \odot p = (ca_0) + (ca_1) x^1 + \cdots + (ca_n) x^n$

Lemma: $(P, \oplus, \odot)$ is a vector space

Let $V = P$, WTS:

$\forall \vec{u}, \vec{v}, \vec{w} \in V$, $\forall c, d \in \mathbb{R}$,

1. V is closed under addition i.e. $\vec{u} \oplus \vec{v} \in V$
2. $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$
3. $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$
4. $\exists \vec{0} \in V$, $\vec{u} \oplus \vec{0} \in V$
5. $\exists -\vec{u} \in V$, $\vec{u} \oplus (-\vec{u}) = \vec{0}$
6. $c \odot \vec{u} \in V$ ← closed under scalar mult
7. $c \odot (\vec{u} \oplus \vec{v}) = c \odot \vec{u} + c \odot \vec{v}$
8. $(c + d) \odot \vec{u} = c \odot \vec{u} + d \odot \vec{u}$
9. $c \odot (d \odot \vec{u}) = (cd \odot \vec{u})$
10. $1 \odot \vec{u} = \vec{u}$

Proof [omitted].

Ex. Another vector space

Let $\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} \}$

Let $f, g \in \mathcal{F}$

Define $\oplus, \odot$:

- $(f \oplus g)(x) = f(x) + g(x)$
- $(c \odot f)(x) = c \cdot f(x)$

Then can prove all those props