Lec 6

More properties of vector spaces

Let V be a vector space (i.e. $(V, +, \cdot)$). Then:

1. $0 \cdot u = \vec{0}$ ← $\vec{0}$ is the zero vector in V
2. $c \cdot \vec{0} = \vec{0}$
3. $(-1)\vec{u} = -\vec{u}$
4. $c \cdot \vec{u} = \vec{0} \Rightarrow (c = 0 \vee u = \vec{0})$

Proof

1. $0 \cdot \vec{u} = 0 \cdot \vec{u} + \vec{0} = 0 \cdot \vec{u} + (\vec{u} - \vec{u}) = (0 \cdot \vec{u} + \vec{u}) + (-\vec{u}) = (0 \cdot \vec{u} + 1 \cdot \vec{u}) + (-\vec{u})$
$= (0+1) \cdot \vec{u} + (-\vec{u}) = \vec{u} - \vec{u} = 0.$

A few more other vector spaces ← A set with structure

Seen: $\mathbb{R}^n$, $\mathcal{P}$ (polynomials), $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$

* $\mathcal{P}_n$ = {polynomials with degree less than or equal to n}
* $\mathcal{C}^0(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous $\}$
* $\mathcal{C}^1(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid f$ differentiable and has continuous derivative $\}$

Linear Transformation ← Map between vector spaces that preserve structure

Let V, W be vector spaces. a linear transformation $T: V \to W$ for all $\vec{u}, \vec{v} \in V$ and $c \in \mathbb{R}$ s.t.

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(c \cdot \vec{u}) = c \cdot T(\vec{u})$

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ via $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 + 3x_2 \end{bmatrix}$. Show it's a linear transformation

$$\left[\begin{aligned} T(\vec{u} + \vec{v}) &= T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} (u_1 + v_1) + (u_2 + v_2) \\ (u_1 + v_1) - (u_2 + v_2) \\ 2(u_1 + v_1) + 3(u_2 + v_2) \end{bmatrix} \\ \cdots &= T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T(\vec{u}) + T(\vec{v}) \end{aligned}\right]_1 \qquad 2\left[\text{ omitted } \right.$$

Ex. Let $D: \mathscr{C}^1(\mathbb{R}) \to \mathscr{C}^0(\mathbb{R})$ via $D(f) = f'$. Show D satisfies 1, 2.

1. $D(f \oplus g)(x) = (f \oplus g)'(x)$
$$= \frac{d}{dx}((f \oplus g)'(x))$$
$$= \frac{d}{dx}(f(x) + g(x))$$
$$= f'(x) + g'(x)$$
$$= (D(f) \oplus D(g))(x)$$

2. $D(c \odot f)(x) = (c \odot f)'(x)$
$$= \frac{d}{dx}(c f(x))$$
$$= c \odot \frac{d}{dx}(f(x))$$
$$= c \odot D(f)(x)$$

Transformation and standard basis vectors

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^3$ is L.T. (linear transformation) and

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$T(e_1) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Then $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
$$= T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$= x\,T(e_1) + y\,T(e_2).$$
← Completely determined by $T(e_1)$ and $T(e_2)$.

Matrix way (brief look)

A be $m \times n$ matrix, $v \in \mathbb{R}^n$, then

$$Av = v_1 \begin{bmatrix} | \\ A_{-1} \\ | \end{bmatrix} + v_2 \begin{bmatrix} | \\ A_{-2} \\ | \end{bmatrix}$$