Lec 12

- Midterm on Wed
  - Review session 18:00 - 19:30
  - One sheet of note allowed

Elementary Matrix

:= matrix you get by doing <u>one</u> elementary row operation.

Ex. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

What happens when we mul some matrix with an elem matrix?

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -3c & -3d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2a+c & 2b+d \end{bmatrix}$$

Mul.ing matrix with elem matrix performs the operation that created the elem matrix

* Notice we can always undo elem. matrix. But then:

$$E_2(E_1A) = A \Rightarrow E_2E_1 = I \Rightarrow E_2 = E_1^{-1}.$$

So all elem. matrix invertible and their inverse is elem. matrix

("fundamental theorem of invertible matrix")

Theorem:

i.e. either all true xor all false

Let $A_{n \times n}$. Then the following are equivalent (TFAR)

(a) A^{-1} exists

(b) $Ax = b$ has unique solution $\forall b \in \mathbb{R}^n$

(c) $Ax = \vec{0}$ has only the trivial solution

(d) Reduced row echelon form of A is I_n

(e) A is a product of elem. matrices

instead of proving $\Leftrightarrow$, we can prove this:

$$a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow a$$

Proof

(a) $\Rightarrow$ (b) Already proven. In short $x = A^{-1}b$ is unique solution.

(b) $\Rightarrow$ (c) $\vec{0} \in \mathbb{R}^n$. So $Ax = \vec{0}$ unique solution. But notice $x = \vec{0}$ is a solution, so $x = \vec{0}$ is the unique solution.

(c) $\Rightarrow$ (d) $Ax = \vec{0}$ can be solved by reducing the aug. matrix

$$\left[\begin{array}{c|c} A & \vec{0} \end{array}\right].$$

But since solution was unique, there's no free variable, the reduced row echelon form must look like:

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & \vec{0} \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \end{array}\right] = \left[\begin{array}{c|c} I_n & \vec{0} \end{array}\right].$$

But then if we take A and put it in reduced row echelon form, the $\vec{0}$ column didn't do anything so we also get I_n.

(d) $\Rightarrow$ (e) If $\overbrace{\text{RREF}}^{\text{reduced row echelon form}}(A) = I_n$, then $E_N \ldots E_2 E_1 A = I_n$

$$\Rightarrow \quad E_N^{-1} E_N \ldots E_2 E_1 A = E_N^{-1} I_n$$

$$\cdots \Rightarrow \quad E_1^{-1} E_2^{-1} \ldots E_N^{-1} E_N \ldots E_2 E_1 A = E_1^{-1} E_2^{-1} \ldots E_N^{-1} I_n$$

$$A = E_1^{-1} E_2^{-1} \ldots E_N^{-1}$$

So A is product of elem. matrices

(e) $\Rightarrow$ (a) Suppose $A = E_1 E_2 \ldots E_M$. Then:

$$E_M^{-1} \ldots E_2^{-1} E_1^{-1} A = E_M^{-1} \ldots E_2^{-1} E_1^{-1} E_2 E_1 \ldots E_M$$

$$E_M^{-1} \ldots E_2^{-1} E_1^{-1} A = I_n$$

$$\text{So } A^{-1} = E_M^{-1} \ldots E_2^{-1} E_1^{-1}$$

Gauss-Jordan Elimination

Let $A_{n\times n}$

Then stick it with a I_n

$$\underset{n \times 2n}{\left[\begin{array}{c:c} A & I_n \end{array}\right]}$$

Lemma:

$$M_{n\times n}\left[\begin{array}{c:c} A & I_n \end{array}\right] = M\left[\begin{array}{c:c} c_1 \cdots c_n & e_1 \cdots e_n \end{array}\right]$$

$$= \left[\begin{array}{c:c} Mc_1 \cdots Mc_n & Me_1 \cdots Me_n \end{array}\right] = \left[\begin{array}{c:c} MA & MI_n \end{array}\right]$$

So we can

$$E_m^{-1} \cdots E_1^{-1}\left[\begin{array}{c:c} A & I_n \end{array}\right] = \left[\begin{array}{c:c} A^{-1}A & A^{-1}I_n \end{array}\right] = \left[\begin{array}{c:c} I_n & A^{-1} \end{array}\right]$$

there we go