Lec 14

Recall properties of determinants from last lec.

* How elem. row ops affect det

since det is L.T. from row when others are fixed.

1. Interchange two rows → mul det by -1
2. Mul one row by non-zero const → mul by that constant
3. Add mul of one row to another → mul by 1

\# Property 6: If $A_{n \times n}$ has a row of zeros, $\det A = 0$

Proof: we can add another row to the zeros row without changing $\det A$ then we have repeated rows $\Rightarrow \det A = 0$
(edge case: $[0]$ in which we can't swap rows, so use L.T. argument)

\# Property 7: If $A_{n \times n}$ is upper triangular, then $\det A$ is product of entries on the main diagonal

$$\det \begin{bmatrix} d_1 & & * \\ & d_2 & \\ 0 & & \ddots & \\ & & & d_n \end{bmatrix} = \prod_{i=1}^{n} d_i$$

Proof

Case 1: $d_1, \ldots, d_n \neq 0$. Do: add mul of each row to the row above to get

$$A' = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & \\ & & & d_n \end{bmatrix}$$

So $\det A = \det^{-1} A'$

$$= d_1 \det \begin{bmatrix} 1 & & 0 \\ & d_2 & \\ 0 & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$= d_1 d_2 \det \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & d_n \end{bmatrix}$$

$$= d_1 \cdots d_n \det \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$= d_1 \cdots d_n$$

Case 2: some d_i is 0. Then add mul of row to another to get a row of all zeros

$$A' = \begin{bmatrix} d_1 & & 0 \\ & 0 & \\ 0 & & d_n \end{bmatrix}$$

We know $\det A' = 0 = \det A$

$\Rightarrow \det A = d_1 \cdots d_n$

Ex.

$$A = \begin{bmatrix} 0 & 2 & -4 \\ 3 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \qquad \det A = (-1)\begin{bmatrix} 1 & 4 & 5 \\ 3 & 0 & -3 \\ 0 & 2 & -4 \end{bmatrix} = (1)(-1)\begin{bmatrix} 1 & 4 & 5 \\ 0 & -12 & -18 \\ 0 & 2 & -4 \end{bmatrix}$$

$$= (-1)(1)(-1)\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & -4 \\ 0 & -12 & -18 \end{bmatrix} = (2)(-1)(1)(-1)\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & -12 & -18 \end{bmatrix}$$

$$= \underbrace{(1)(2)(-1)(1)(-1)}\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & -42 \end{bmatrix} = -84$$

interestingly, these are determinants of the elem matrices for our operations

* $\operatorname{rref}(A) = I_n \Leftrightarrow \det(A) \neq 0 \Leftrightarrow \exists A^{-1} \Rightarrow$ "A is <u>non-singular</u>"

* $\operatorname{rref}(A) \neq I_n \Leftrightarrow \det(A) = 0 \Leftrightarrow \nexists A^{-1} \Rightarrow$ "A is <u>singular</u>"