Lec 15

Determinant of prod of matrix

<u>Lemma</u>: let E be elem matrix and $A_{n\times n}$. Then $\det(EA) = \det(E)\det(A)$

<u>Proof</u> case on how we got E.

If row exchange, $\det E = -\det I_n = -1$

$\Rightarrow \det(EA) = -1 \det A$
$= \det(E)\det(A)$

If mul row: $\det E = c \det I_n = c$ for some constant c

$\Rightarrow \det(EA) = c \det A$
$= \det E \det A$

If add mul of row: $\det E = \det I_n = 1$

$\Rightarrow \det(EA) = \det A$
$= \det E \det A$

* Proposition 9

If $A_{n\times n}$ and $B_{n\times n}$, $\det(AB) = \det A \det B$

<u>Proof</u> If A singular, $\det A = 0 \Rightarrow \det A \det B = 0$

Also, A singular $\Rightarrow$ AB singular $\Rightarrow \det AB = 0$

If A invertible, then $A = E_1 E_2 \dots E_N$ for elem m.s $E_{1\dots N}$

Then $\det(AB) = \det(E_1 \cdots E_N B)$
$= \det E_1 \det(E_2 \cdots E_N B)$
$\cdots = \det(E_1 \cdots E_N) \det B$
$= \det A \det B$

Det of transpose

* Property 10: let $A_{n\times n}$. $\det(A^T) = \det(A)$ [Proof omitted]

<u>Implications</u>: properties about row of matrix also apply to row!

General recursive formula

Base case (2×2 mx.)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} - \begin{vmatrix} c & d \\ 0 & b \end{vmatrix}$$

$$= ad - cb$$

3×3 case :

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ e & d & f \\ h & g & i \end{vmatrix} + \begin{vmatrix} c & 0 & 0 \\ f & d & e \\ i & g & h \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ e & d & 0 \\ h & g & i \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ 0 & 0 & f \\ h & g & i \end{vmatrix} + \begin{vmatrix} c & 0 & 0 \\ f & d & 0 \\ i & g & h \end{vmatrix} + \begin{vmatrix} c & 0 & 0 \\ 0 & 0 & d \\ i & g & h \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & 0 & 0 \\ 0 & f & 0 \\ g & i & h \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ e & d & 0 \\ h & g & i \end{vmatrix} + \begin{vmatrix} b & 0 & 0 \\ 0 & f & 0 \\ h & i & g \end{vmatrix} + \begin{vmatrix} c & 0 & 0 \\ f & d & 0 \\ i & g & h \end{vmatrix} - \begin{vmatrix} c & 0 & 0 \\ 0 & d & 0 \\ i & h & g \end{vmatrix}$$

$$= aei - afh - bdi + bfg + cdh - cdg$$

Aside

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

$$\left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right. = aei + bfg + cdh - ceg - afh - bdi$$

⚠ Doesn't generalise to higher dim!

To be continued ...