

Lec 15

Determinant of prod of matrix

Lemma: let E be elem matrix and $A_{n \times n}$. Then $\det(EA) = \det(E)\det(A)$

Proof case on how we get E .

If row exchange, $\det E = -\det I_n = -1$

$$\begin{aligned}\Rightarrow \det(EA) &= -1 \det A \\ &= \det(E)\det(A)\end{aligned}$$

If mul row: $\det E = c \det I_n = c$ for some constant c

$$\begin{aligned}\Rightarrow \det(EA) &= c \det A \\ &= \det E \det A\end{aligned}$$

If add mul of row: $\det E = \det I_n = 1$

$$\begin{aligned}\Rightarrow \det(EA) &= \det A \\ &= \det E \det A\end{aligned}$$

* Proposition 9

If $A_{n \times n}$ and $B_{n \times n}$, $\det(AB) = \det A \det B$

Proof If A singular, $\det A = 0 \Rightarrow \det A \det B = 0$

Also, A singular $\Rightarrow AB$ singular $\Rightarrow \det AB = 0$

If A invertible, then $A = E_1 E_2 \dots E_N$ for elem m.s E_1, \dots, E_N

$$\begin{aligned}\text{Then } \det(AB) &= \det(E_1 \dots E_N B) \\ &= \det E_1 \det(E_2 \dots E_N B) \\ &\dots = \det(E_1 \dots E_N) \det B \\ &= \det A \det B\end{aligned}$$

Det of transpose

* Property 10: let $A_{n \times n}$. $\det(A^T) = \det(A)$ [Proof omitted]

Implications: properties about row of matrix also apply to row!

General recursive formula

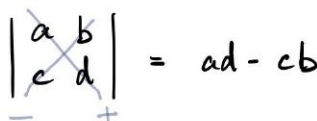
Base case (2x2 mx.)


$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \\ &= \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} - \begin{vmatrix} c & d \\ 0 & b \end{vmatrix} \\ &= ad - cb \end{aligned}$$

3x3 case :

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix} \\ &= \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ e & d & f \\ h & g & i \end{vmatrix} + \begin{vmatrix} c & 0 & 0 \\ f & d & e \\ i & g & h \end{vmatrix} \\ &= \begin{vmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ g & h & i \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ e & d & 0 \\ h & g & i \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ 0 & 0 & f \\ h & g & i \end{vmatrix} + \begin{vmatrix} c & 0 & 0 \\ f & d & 0 \\ i & g & h \end{vmatrix} + \begin{vmatrix} c & 0 & 0 \\ 0 & 0 & d \\ i & g & h \end{vmatrix} \\ &= \begin{vmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{vmatrix} - \begin{vmatrix} a & 0 & 0 \\ 0 & f & 0 \\ g & i & h \end{vmatrix} - \begin{vmatrix} b & 0 & 0 \\ e & d & 0 \\ h & g & i \end{vmatrix} + \begin{vmatrix} b & 0 & 0 \\ 0 & f & 0 \\ h & i & g \end{vmatrix} + \begin{vmatrix} c & 0 & 0 \\ f & d & 0 \\ i & g & h \end{vmatrix} - \begin{vmatrix} c & 0 & 0 \\ 0 & d & 0 \\ i & h & g \end{vmatrix} \\ &= aei - afh - bdi + bfg + cdh - cdg \end{aligned}$$

Aside

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$


$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$$


⚠ Doesn't generalise to higher dim!
To be continued ...