$\boxed{\text{Lec 16}}$

Recursive formula for det of matrix (cont.)

Lemma: If $A_{n\times n} = \left[\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & B & \\ * & & & \end{array}\right]$, $\det A = a \det B$

Proof: If $a=0$, $\det A = 0 = 0 \det B = 0 = a \det B$ ✓

Else $(a \neq 0)$:

We can use add mul of row to another to get:

$$\hat{A} = \left[\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right] \quad \text{with } \det \hat{A} = \det A$$

Let $C_{(n-1)\times(n-1)}$

Lemma:

$$\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & C & \\ 0 & & & \end{array}\right]\left[\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right] = \left[\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & CB & \\ 0 & & & \end{array}\right]$$

Proof:

$$\left[\begin{array}{c|c} [1][a] + [0 \cdots 0]\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} & [1][0 \cdots 0] + [0 \cdots 0]B \\ \hline \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}[a] + C\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}[0 \cdots 0] + CB \end{array}\right]$$

We can write $\operatorname{rref}(B) = E_N \cdots E_1 B$

Then $\det(\operatorname{rref}(B)) = \det(E_N) \cdots \det(E_1)\det(B)$

For each $E_{(n-1)(n-1)}$, we can build

$\tilde{E} = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & E & \\ 0 & & & \end{array}\right]$, which is also an elem mx.
And we did same op to get as E
So $\det E = \det \tilde{E}$

Then $\tilde{E}_N \cdots \tilde{E}_1 \hat{A} = \left[\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \operatorname{rref}(B) & \\ 0 & & & \end{array}\right]$

If $\operatorname{rref}(B)$ has a row of 0s, $\det B = 0$, so $\det(\tilde{E}_N \cdots \tilde{E}_1 \hat{A}) = 0$
$\Rightarrow \det \hat{A} = 0 \Rightarrow \det A = 0 \Rightarrow \det A = a \det B$ ✓

Else: (rref(B) doesn't have a row of 0s)

So $\text{rref}(B) = I_{(n-1)} \Rightarrow \det(E_N \cdots E_1 B) = 1$

Then $\tilde{E}_N \cdots \tilde{E}_1 \hat{A} = \left[\begin{array}{c|ccc} a & 0 & \cdots & 0 \\ \hline 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{array}\right] \Rightarrow \det(\tilde{E}_N \cdots \tilde{E}_1 \hat{A}) = a$

So $\underbrace{\det(E_N \cdots E_1)}\det B = 1$

$\underbrace{\det(\tilde{E}_N \cdots \tilde{E}_1)}\det \hat{A} = a$ ⎵ same

$\Rightarrow \quad a \det B = \det \hat{A} = \det A \qquad \square$

Now consider:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}$$

Let A_{ij} be A with i^{th} row and j^{th} col removed

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & \cancel{\;} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ \approx & & & & \approx \\ \vdots & & & & \vdots \\ a_{1n} & \cdots & \cancel{\;} & \cdots & a_{nn} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{vmatrix}$$

keep order

$$= \left|\begin{array}{c|ccc} a_{11} & 0 & \cdots & 0 \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right| + (-1)^1 \left|\begin{array}{c|ccc} a_{12} & 0 & \cdots & 0 \\ \hline a_{22} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n1} & \cdots & a_{nn} \end{array}\right| + \cdots + (-1)^{n-1} \left|\begin{array}{c|ccc} a_{1n} & 0 & \cdots & 0 \\ \hline a_{2n} & a_{21} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & & \vdots \\ a_{nn} & a_{n1} & \cdots & a_{n(n-1)} \end{array}\right|$$

$$= \left|\begin{array}{c|ccc} a_{11} & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & A_{11} & \\ * & & & \end{array}\right| + (-1)^1 \left|\begin{array}{c|ccc} a_{12} & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & A_{12} & \\ * & & & \end{array}\right| + \cdots + (-1)^{j-1} \left|\begin{array}{c|ccc} a_{1n} & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & A_{1j} & \\ * & & & \end{array}\right| + \cdots +$$

$$(-1)^{n-1} \left|\begin{array}{c|ccc} a_{1n} & 0 & \cdots & 0 \\ \hline * & & & \\ \vdots & & A_{1n} & \\ * & & & \end{array}\right|$$

$$= a_{11}|A_{11}| + (-1)^1 a_{12}|A_{12}| + \cdots + (-1)^{j-1} a_{1j}|A_{1j}| + \cdots + (-1)^{n-1} a_{1n}|A_{1n}|$$

$$= \sum_{j=1}^{n} (-1)^{j-1} a_{1j} |A_{1j}|$$ ← "Laplace expansion across first row"