Lec 18

Eigenvalues & Eigenvectors

* Useful for computation, visualising L.T., solving some diffEQ, hedging bond portfolios, etc.

if we allow $\vec{0}$, any λ works...? and we get non bijection between eigenval & eigenvec...?

Def: If $A_{n\times n}$ and $x \in \mathbb{R}^n$, $x \neq \vec{0}$ s.t. $Ax = \lambda x$ f.s. $\lambda \in \mathbb{R}$.
Then λ is an eigenvalue of A and x is an eigenvector of A with that eigenvalue.

German, means self reference?

Eigenvector equation: $Ax = \lambda x$

Finding x:

$$Ax - \lambda(I_n x) = 0$$
$$Ax - (\lambda I_n)x = 0$$
$$\underbrace{(A - \lambda I_n)}x = 0$$

must not be invertible, else $x = 0$, not what we want.

$$\Rightarrow \det(A - \lambda I_n) = 0$$

then solve for λ

Ex. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad A - \lambda I_2 = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

$$\det(A - \lambda I_2) = (3-\lambda)^2 - 1^2 = 0$$
$$3 - \lambda = \pm 1$$
$$\lambda_1 = 2, \ \lambda_2 = 4$$

FTI gurentees each correspond to some x

multiple eigenval possible.

Solve: $(A - 2I_2)x = 0$, $\quad (A - 4I_2)x = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad x = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

but $x \neq 0$, so $\qquad$ but $x \neq 0$, so

$x = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with $t \neq 0$. $\qquad x = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $t \neq 0$

Def: $\det(A - \lambda I_n)$ is characteristic polynomial for $A_{n\times n}$

always polynomial with degree n

Thm if A triangular, entries on main diagonal are the eigenvalues.

Pf Consider

$$A = \begin{bmatrix} d_1 & & * \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{bmatrix}$$

$$\det(A - \lambda I_n) = \begin{vmatrix} d_1 - \lambda & & & * \\ & d_2 - \lambda & & \\ & & \ddots & \\ 0 & & & d_n - \lambda \end{vmatrix}$$

$$= (d_1 - \lambda)(d_2 - \lambda) \cdots (d_n - \lambda).$$

$\det(A - \lambda I_n) = 0 \Rightarrow \lambda = d_1, d_2, \ldots,$ or d_n

Thm $A_{n \times n}$ is invertible iff 0 is not an eigenval

Pf A invertible iff $\det(A) \neq 0$.
0 not eigenval of A $\Leftrightarrow \det(A - 0I_n) \neq 0$
$\Leftrightarrow \det(A) \neq 0$

Thm Let $A_{n \times n}$ with eigenval λ and corresponding eigenvec x. Then:

1. $\forall n \in \mathbb{Z}^+$, λ^n is e-val of A^n w~ e-vec x
2. If $\exists A^{-1}$, $1/\lambda$ is e-val of A^{-1} w~ e-vec x
3. If $\exists A^{-1}$, $\forall n \in \mathbb{Z}^+$, λ^{-n} is e-val of A^{-n} w~ e-vec x

Pf1 BC $(n=1)$ $A^1 x = Ax = \lambda x$
IC $(n \geq 1)$
IH $A^n x = \lambda^k x$
IS $A^{n+1} x = A(A^n x) = A(\lambda^n x) = \lambda^n (Ax) = \lambda^n \lambda x = \lambda^{n+1} x$

Pf2 $Ax = \lambda x$
$A^{-1} A x = A^{-1} \lambda x$
$Ix = \lambda (A^{-1} x)$ not zero since $\exists A^{-1}$
$A^{-1} x = \frac{1}{\lambda} x$ as reqed