Lec 20

Span & eigen

Def Let $v_1, \dots, v_k$ in a vector space V. Take some linear combo the span of these vectors is $\{ c_1 v_1 + \cdots + c_k v_k \mid c_1, \dots, c_k \in \mathbb{R} \}$

Lemma if $v_1, \dots, v_k$ are e-vecs of A with same e-val λ, then any non-zero linear combo of $v_1, \dots, v_k$ is also a e-vec of A with e-val λ.

Proof

Suppose $c_1 v_1 + \cdots + c_k v_k \neq 0$

then $A(c_1 v_1 + \cdots + c_k v_k) = c_1 A v_1 + \cdots + c_k A v_k$

$= \lambda (c_1 v_1 + \cdots + c_k v_k)$

so $c_1 v_1 + \cdots + c_k v_k$ is e-vec by def

Def Let $A_{n \times n}$ with e-val λ, then the set of corresponding e-vec together with $\vec{0}$ is the λ-eigenspace, denoted E_λ.

Ex (using A from last lec) $E_0 = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$

Thm E_λ is a vector space

Subspace

Def A non-empty subset W of a vector space V is a subspace of V if W is a vector space with the same $\oplus$ and $\odot$ as V.

Ex. $W = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \in \mathbb{R}^3 \mid x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$

Thm If V is vec space and $W \subseteq V$, $W \neq \emptyset$, then W is subspace of V iff:

1. Closed under addition
2. Closed under scaler mul

Pf WTS W satisfies the 10 axioms

1. assumed

2. addition commutative in $V \Rightarrow$ in W too
3. ... associative - - -
4. $\exists w \in W$, $\vec{0} = 0w$, but W closed under scaler mul, so $\vec{0} = 0w \in W$
5. $-w = (-1)w \in W$ since W closed under scaler mul
6. assumed
7.
⋮ — all true in V so also true in W.
10.

* Trivial subspaces:
- $\{\vec{0}_V\}$ is always subspace of V
- V is always subspace of V

<u>Thm</u> Let V be vector space with vecs $v_1, \dots, v_k \in V$, then
- $\text{span}(v_1, \dots, v_k)$ is subspace of V
- Any subspace that contain $v_1, \dots, v_k$ contains $\text{span}(v_1, \dots, v_k)$

Viz. $\text{span}(v_1, \dots, v_k)$ is the smallest subspace that contains $v_1, \dots, v_k$

Pf
Let $W = \text{span}(v_1, \dots, v_k)$
Then $v_1 \in W \Rightarrow W \neq \phi$ as req
Also, suppose $c_1v_1 + \dots + c_kv_k \in W$, $d_1v_1 + \dots + d_kv_k \in W$,
then $c_1v_1 + \dots + c_kv_k + d_1v_1 + \dots + d_kv_k$
$= (c_1+d_1)v_1 + \dots + (c_k+d_k)v_k \in W$ as req
Finally, $r(c_1v_1 + \dots + c_kv_k)$
$= (rc_1)v_1 + \dots + (rc_k)v_k \in W$ as req

* Some interesting subspaces for $A_{m \times n}$

Recall $A = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & r_1^T & - \\ & \vdots & \\ - & r_m^T & - \end{bmatrix}$

- $\text{col}(A) = \text{span}(c_1, \dots, c_n)$, which is a subspace of $\mathbb{R}^n$
- $\text{row}(A) = \text{span}(r_1, \dots, r_m)$ - - - - - - $\mathbb{R}^m$