Lec 21

Some more defs

<u>Def</u> If $W = \text{span}(v_1, \cdots, v_k)$, we say
- $v_1, \cdots, v_k$ <u>span</u> W
- $\{v_1, \cdots, v_k\}$ is a <u>spanning</u> set of W

<u>Def</u> The <u>null space</u> of $A_{m \times n}$ is $\text{nul}(A) = \{ x \in \mathbb{R}^n \mid Ax = \vec{0} \}$

<u>Claim</u> $\text{nul}(A)$ is a subspace of $\mathbb{R}^n$

<u>Proof</u> First $\text{nul}(A) \neq \emptyset$ since $A\vec{0}_{\mathbb{R}^n} = \vec{0}_{\mathbb{R}^n} \Rightarrow \vec{0}_{\mathbb{R}^n} \in \text{nul}(A)$

Now let $\vec{x}, \vec{y} \in \text{nul}(A)$. WTS $\vec{x}+\vec{y} \in \text{nul}(A)$.
Well $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = 0 \Rightarrow \vec{x}+\vec{y} \in \text{nul}(A)$

Finally let $c \in \mathbb{R}$. WTS $c\vec{x} \in \text{nul}(A)$
Well $A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0} \Rightarrow c\vec{x} \in \text{nul}(A)$ □

Linear Independence

<u>Def</u> $\{v_1, \dots, v_k\}$ in vector space V is <u>linearly dependent</u> if

$$\exists c_1, \dots, c_k \in \mathbb{R}, \big(\underbrace{\neg \forall c \in \{c_1, \dots, c_k\},\ c = \vec{0}}_{\text{none of } \vec{c} \text{ is } \vec{0}},\ \underbrace{(c_1 v_1 + \cdots + c_k v_k = \vec{0})}_{\text{but can combine to } \vec{0}}\big)$$

<u>Def</u> $c_1 v_1 + \cdots + c_k v_k = \vec{0}$ is a <u>linear dependence relation</u>

- If $c_1 = \cdots = c_k = 0$, it's a <u>trivial</u> lin. dep. rel.
- Else it's <u>non-trivial</u>

<u>Ex</u> Find all lin. dep. rel. for $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$

Want: $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} = 0$

Solve $\rightsquigarrow (c_1, c_2) = 0$
It's only the trivial one, so $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$ are lin. indep.

<u>Ex</u> Find all lin. dep. rel. for $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

Solve $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -c_3 \\ c_3 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

So we have non-trivial sols if $c_3 \neq 0$, so set is lin. dep.

<u>Thm</u> Let $v_1, \dots, v_n \in \mathbb{R}^m$ and let

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}.$$

Then $\{v_1, \dots, v_n\}$ is lin. indep. iff $A\vec{x} = \vec{0}$ has only the trivial sol

<u>Proof</u> just by defs.

Notice if $n = m$, $A\vec{x} = \vec{0}$ has only the trivial sol $\Leftrightarrow \exists A^{-1}$.

<u>Thm</u> $\{v_1, \dots, v_k\}$ is lin. dep. iff some $v_i \in \{v_1, \dots, v_k\}$ can be written as a linear combo of $\{v_1, \dots, v_k\} \setminus \{v_i\}$

<u>Thm</u> If $\underline{m > n}$, then $\{v_1, \dots, v_m\} \subseteq \mathbb{R}^n$ is lin. dep.
↳ think of as no more dimension to point in after using all n.

<u>Proof</u>

Well we want sol. $\left.\left[\begin{array}{ccc|c} | & & | & | \\ v_1 & \cdots & v_m & 0 \\ | & & | & | \end{array}\right]\right\} n$

But at most we have: $\left[\begin{array}{ccccc|c} 1 & & 0 & | & | & \\ & \ddots & & * & * & 0 \\ 0 & & 1 & | & | & \end{array}\right]$

↑ We always have free vars since we can have max of n leading out of m vars

<u>Def</u> A set $B = \{b_1, \dots, b_n\}$ in vec. space V is a <u>basis</u> if

1. B spans V
2. B is lin. indep.