Lec 22

Theorems about basis

<u>Def</u> Recall basis of v.s. V := set of lin. indep. vecs that spans V

<u>Thm</u> If $B = \{b_1, \dots, b_n\}$ is basis of v.s. V, then $v \in V$ can be written as a unique linear combo $v = c_1 b_1 + \cdots + c_n b_n$

<u>Proof</u> Let $v \in V$, since B spans V, $v = c_1 b_1 + \cdots + c_n b_n$
Suppose $v = d_1 b_1 + \cdots + d_n b_n$

Then $\vec{0} = v - v = (d_1 - c_1) b_1 + \cdots + (d_n - c_n) b_n$
$\Rightarrow$ $d_1 - c_1 = 0$, $\cdots$, $d_n - c_n = 0$ since B lin. indep.
$\Rightarrow$ $d_1 = c_1$, $\cdots$, $d_n = c_n$

<u>Thm</u> If $B = \{b_1, \dots, b_n\}$ spans v.s. V, then $A = \{v_1, \dots, v_m\} \subseteq V$ with $|A| > |B|$ in lin dep.

<u>Proof</u> WTS there is a non-trivial $c_1 v_1 + \cdots + c_m v_m = \vec{0}_V$

Since B spans V,

$v_1 = a_{11} b_1 + \cdots + a_{1n} b_n$
$\vdots$
$v_m = a_{m1} b_1 + \cdots + a_{mn} b_n$

Then $c_1 v_1 + \cdots + c_m v_m$
$= c_1 (a_{11} b_1 + \cdots + a_{1n} b_n) + \cdots + c_m (a_{m1} b_1 + \cdots + a_{mn} b_n)$
$= (c_1 a_{11} + \cdots + c_m a_{m1}) b_1 + \cdots + (c_1 a_{1m} + \cdots + c_m a_{mn}) b_n = \vec{0}$

But then
$c_1 a_{11} + \cdots + c_m a_{m1} = 0$, $\cdots$, $c_1 a_{1m} + \cdots + c_m a_{mn} = 0$
If we solve for c_j, there are m unknowns and n equations, so there will be free variables.

So we can choose some c_j to be non zero to make $c_1 v_1 + \cdots + c_m v_m$ have non-trivil solution.

Thm If $B = \{b_1, \dots, b_n\}$ is lin indep. and $m < n$,
then $\{v_1, \dots, v_m\} \subseteq W = \text{span}(B)$ cannot span W.

Proof suppose $\text{span}(v_1, \dots, v_m) = \text{span}(b_1, \dots, b_n)$
but then $\{b_1, \dots, b_n\}$ is lin dep. since $n > m$,
that contradicts B being lin indep.

Thm If $B = \{b_1, \dots, b_n\}$ is a basis for v.s. V, then any other basis for V has to have n vectors

Proof
more than $n \Rightarrow$ not lin. indep.
less than $n \Rightarrow$ can't span

Def The number of vecs in a basis is the <u>dimension</u> of the v.s.

Spaces

Thm Doing elem. row ops does not change row space of matrix