$\boxed{\text{Lec 23}}$

Midterm next class

Continue from last lec

<u>Thm</u> Doing elem. row ops does not change row space of matrix
More formally:
if A' is obtained from A by elem row ops, row A = row A'

<u>Proof</u>
We case elem row ops on
(i) Interchange rows [exercise]
(ii) Mul row [exercise]
(iii) Add mul of row j to row i

$$A = \begin{bmatrix} \vdots \\ r_i^T \\ \vdots \\ r_j^T \\ \vdots \end{bmatrix} \rightsquigarrow A' = \begin{bmatrix} \vdots \\ r_i^T + t r_j^T \\ \vdots \\ r_j^T \\ \vdots \end{bmatrix}$$

Let $v \in \text{row}(A)$, WTS $v \in \text{row}(A')$

$$\begin{aligned} v &= \cdots + c_i r_i + \cdots + c_j r_j + \cdots \\ &= \cdots + c_i r_i + \cdots + c_j r_j + \cdots + c_i t r_j - c_i t r_j \\ &= \cdots + c_i \underline{(r_i + t r_j)} + \cdots + (c_j - c_i t) \underline{r_j} + \cdots \\ &\in \text{row}(A') \end{aligned}$$

Let $v \in \text{row}(A')$ WTS $v \in \text{row}(A)$ [simili]

<u>Crlr</u> $\text{row}(A) = \text{row}(\text{rref}(A))$

<u>Fact</u> The non-zero rows in rref(A) form a lin indep. set.

Ex. $A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

these lin indep

↳ So we can find a basis by taking rref

↳ $\dim(\text{row}(A))$ = num of leading 1s in rref(A)

Thm Doing elem row ops on A changes row(A)

Counterexample: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

But the linear dependence rel among the cols doesn't change.

$R = \text{rref}(A) \Leftrightarrow \text{rref}\left[A \,\vdots\, \vec{0} \right] = \left[R \,\vdots\, \vec{0} \right]$

$A\vec{x} = \vec{0} \Leftrightarrow R\vec{x} = \vec{0}$

Supp $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$, $R = \begin{bmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{bmatrix}$

Consider

$c_1 v_1 + \cdots + c_n v_n = \vec{0}$

Then $A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \Leftrightarrow R\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \Rightarrow c_1 w_1 + \cdots + c_n w_n = \vec{0}$

Fact The cols with leading 1 are a basis for col(R)

$\Downarrow$

Thm Those corresponding cols in A are basis for col(A)

Ex. $A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

basis for col(A)

$\hookrightarrow$ dim(col(A)) = num of leading 1s = dim(row(A))

Basis for null space

Recall: $\text{nul}(M) = \{ x : Mx = \vec{0} \}$

Thm Let $R = \text{rref}(A)$, then $\text{nul}(A) = \text{nul}(R)$

Ex. $A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Solve $R\vec{x} = 0$ yields $\begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} c_4$, and $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right\}$ is span for nul(R)

Ex.

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{solves to} \quad c_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c_6 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

↳ dim (null (A)) = num of free vars in R