Lec 24

* More row, col, null spaces

Consider: $A = \begin{bmatrix} 1 & 2 & -5 & 4 & 54 \\ 0 & 0 & -3 & -75 & -744 \\ 6 & 12 & -24 & 59 & 662 \\ -2 & -4 & 8 & -18 & -204 \end{bmatrix}$, $R = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

↪ so a basis for row(A) is $\{ \ , \ , \ \}$ whatever.

* Same # of pivot ⇒ same dim ↕ Called the <u>rank</u> of matrix

↪ a basis for col(A) is $\{ \ , \ , \ \}$ Always subset of col vectors

Null space $\text{null}(A) = \{ \text{sols to } A\vec{x} = \vec{0} \}$

→ Find vector form for $A\vec{x} = \vec{0}$, we'll get lin. indep. vecs.

$$\begin{aligned} x_1 + 2x_2 + 4x_5 &= 0 \\ x_3 - 2x_5 &= 0 \\ x_4 + 10x_5 &= 0 \end{aligned} \Rightarrow \vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ 2 \\ -10 \\ 1 \end{bmatrix}$$

lin indep.

↪ a basis for null A is $\{ \ , \ \}$

<u>Def</u> <u>Rank</u> of matrix $= \dim(\text{col}(A)) = \dim(\text{row}(A))$
= num of pivots
= num of leading vars

<u>Def</u> <u>Nullity</u> of matrix $= \dim(\text{null}(A))$
= num of free vars
= num of non-pivot cols.

<u>Thm</u> The Rank Theorem: if A has n cols:

$$\text{rank } A + \text{nullity } A = n$$

<u>Thm</u> $\text{rank}(A) = \text{rank}(A^T)$

<u>Proof</u> $\text{rank}(A^T) = \dim(\text{row}(A^T)) = \dim(\text{col}(A))$
$= \text{rank}(A)$

Thm Let $A_{n\times n}$, then the following $\Leftrightarrow$

b. $A\vec{x} = \vec{b}$ has unique sol
c. $A\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as sol
d. $\text{rref}(A) = I_n$
f. $\text{rank}(A) = n$
h. Cols of A are lin indep
i. Cols of A spans $\mathbb{R}^n$
j. Cols of A forms a basis for $\mathbb{R}^n$

Proof using dir of impl. shown '$\rightarrow$'

($f \Rightarrow d$) If rank $A = n$, there are n leading entries in rref $A \Rightarrow$ rref $A = I_n$

($d \Rightarrow c$)
($c \Rightarrow b$) by FToIM

($b \Rightarrow i$) $A\vec{x} = \vec{b}$ has unique sol for all $\vec{b} \in \mathbb{R}^n$, then $A\vec{x}$ being linear combo of cols of $A \Rightarrow$ col $A = \mathbb{R}^n$

($i \Rightarrow f$) col $A = \mathbb{R}^n$ then rank $A = \dim(\text{col} A) = \dim(\mathbb{R}^n) = n$

($c \Leftrightarrow h$) $A\vec{x} = \vec{0} \Rightarrow \vec{x} = 0$

$A = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}$, $A\vec{x} = x_1c_1 + \cdots + x_nc_n = \vec{0} \Rightarrow c_i$ lin indep

Same in reverse

($f \wedge h \Leftrightarrow j$) By def