$\boxed{\text{Lec 25}}$

Updated Fund Thm of Invertible Matrices (putting things together)

<u>Thm</u> Let $A_{n\times n}$, TFAE:

a. $\exists A^{-1}$
b. $Ax = b$ has unique sol
c. $Ax = \vec{0}$ has only trivil sol
d. rref $A = I_n$
c. A is prod of elem matrices
f. rank $A = n$
g. nullity $A = 0$
h. cols of A are lin. indep. — k.
i. cols of A span $\mathbb{R}^n$ — l.
j. cols of A form a basis for $\mathbb{R}^n$ — m.
(k., l., m.: same for rows)
n. det $A \neq 0$
o. 0 is not an e-val for A

Eigenspaces

Recall $Av = \lambda v \Leftrightarrow (A - \lambda I)v = \vec{0} \Leftrightarrow A - \lambda I$ not invertible

Eigenspace $E_\lambda = \{ v \mid (A - \lambda I)v = \vec{0} \} \cup \{\vec{0}\}$

→ Can we relate this eigenspace to one of the subspaces we know?

<u>Thm</u> $E_\lambda = \text{nul}(A - \lambda I)$

Consider: $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ with $\lambda = 3$, $\lambda = 2$

both double root

Basis for E_2: solve $\left[\begin{array}{cccc|c} 2-2 & 0 & 0 & 0 & 0 \\ 0 & 3-2 & 1 & 0 & 0 \\ 0 & 0 & 3-2 & 0 & 0 \\ 0 & 0 & 0 & 2-2 & 0 \end{array}\right]$

yields $\begin{bmatrix} v_1 \\ 0 \\ 0 \\ v_4 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\{ \quad , \quad \}$

Basis for E_3 : solve [omitted]

yields $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Notice $\dim E_2 = 2$ ← matches double root

$\dim E_3 = 1$ ← hmm

Def Algebraic multiplicity of e-val λ_i is the number of times $(\lambda - \lambda_i)$ show up as a factor of $\det(A - I\lambda)$.

$$\det(A - \lambda I) = (\lambda-2)\underbrace{(\lambda-3)(\lambda-3)}_{3 \text{ has multiplicity } 2}(\lambda-2)$$

so does 2

Def Geometric multiplicity of λ_i is $\dim(E_{\lambda_i})$

Notice: We can always find one line of e-vecs, so $1 \leq$ geometric multiplicity and geometric multiplicity $\leq$ algebraic multiplicity

If geo. mult. = alge mult. for all the e-vals, we get basis for $\mathbb{R}^n$ and we can take large power of A, etc.
Else maybe not

Orthogonality

Def Dot product [omitted]

that's an inner product

Properties

1. $x \cdot y = y \cdot x$
2. $x \cdot (y+z) = x \cdot y + x \cdot z$
3. $cx \cdot y = c(x \cdot y)$
4. a. $x \cdot x \geq 0$
   b. $x \cdot x = 0 \Leftrightarrow x = \vec{0}$

Def Inner product : operation in any V.S. that satisfy these
For V.S. V, an inner prod is an op that assigns to any pair of vecs $u, v \in V$ a real num $\langle u, v \rangle \in \mathbb{R}$ s.t. for all $u, v, w \in V$ and $c \in \mathbb{R}$,

1. $\langle u, v \rangle = \langle v, u \rangle$

2. $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

3. $\langle cu, v \rangle = c\langle u, v \rangle$

4a. $\langle u, u \rangle \geq 0$

b. $\langle u, u \rangle = 0 \iff v = 0$

Ex. Let P be V.S. of degree 2 polynomials.

Define $\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, dx$

This satisfies 1-4.