Lec 26

Norm

<u>Def</u> In $\mathbb{R}^n$, the <u>length</u> aka <u>norm</u> of a vec $\vec{x}$ is

$$\|\vec{x}\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

This generalises to V.S. with inner prod $\langle \cdot, \cdot \rangle$

The <u>norm</u> of $v \in$ V.S. V is

$$\|\vec{v}\| = \sqrt{\langle v, v \rangle}$$

<u>Def</u> <u>Inner product space</u> is a V.S. with an inner prod

$$(V, \underbrace{\langle \cdot, \cdot \rangle_V}_{\text{adding additional structure to a vector space}})$$

<u>Notice</u>
- $\|c\vec{u}\| = |c|\,\|\vec{u}\|$
- $\|\vec{u}\| = 0 \Leftrightarrow \sqrt{\langle u, u \rangle} = 0 \Leftrightarrow \langle u, u \rangle = 0 \Leftrightarrow u = \vec{0}_V$
- $\|u + v\| \leq \|u\| + \|v\|$ ← triangle inequality

$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$

$\|u+v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$\langle u+v, u+v \rangle \leq \langle u, u \rangle + 2\|u\|\|v\| + \langle v, v \rangle$

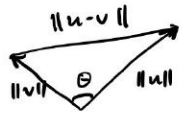
$\langle u,u \rangle + \langle v,u \rangle + \langle u,v \rangle + \langle v,v \rangle \leq \dots$

$2\langle u, v \rangle \leq 2\|u\|\|v\|$

$\langle u, v \rangle \leq \|u\|\|v\|$

└ Cauchy's Inequality

- $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta$



$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

For general I.P.S, we define $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\|\|v\|}$

Def in a I.P.S. , we say that vecs u, v are orthogonal if $\langle u, v \rangle = 0$.

Note $\vec{0}$ is orthogonal to every vector ← $\langle \vec{0}_V, u \rangle = 0$

$\langle 2\vec{0}_V, u \rangle = 2\langle \vec{0}_V, u \rangle = \langle \vec{0}_V, u \rangle$

$\Rightarrow \langle \vec{0}_V, u \rangle = 0$

Def A set of vecs $\{v_1, \dots, v_k\}$ is mutually orthogonal if $\langle v_i, v_j \rangle = 0$ for all $v_i, v_j \in$ the set and $i \neq j$

Thm If $\vec{0} \notin \{v_1, \dots, v_k\}$ and the set is mutually orthogonal implies it's lin indep.

Proof Suppose $c_1 v_1 + \dots + c_k v_k = \vec{0}$

$\langle v_i, c_1 v_1 + \dots + c_k v_k \rangle$

$= c_1 \langle v_i, v_1 \rangle + \dots + c_k \langle v_i, v_k \rangle$

$= c_i \langle v_i, v_i \rangle$

$= c_i \|v_i\|^2 = 0$

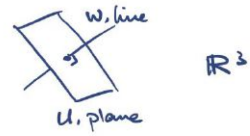
$\Rightarrow c_i = 0$ since $v_i \neq \vec{0}$

Def A set of vecs $\{v_1, \dots, v_k\}$ is <u>orthonormal</u> if they are mutually orthogonal and $\|v_i\| = 1$ for all $v_i \in$ the set.

Ex. standard basis vec

Def Let V be I.P.S. and U, W be subspaces. V and W orthogonal if $\forall u \in U, \forall w \in W, \langle u, w \rangle = 0$

Ex.



Notation We can write $U \perp W$