$\boxed{\text{Lec 27}}$

More theorems on orthogonality

Thm Suppose U, W are subspaces of V with inner prod $\langle \cdot, \cdot \rangle$,
$U = \text{span}(u_1, \dots, u_k)$, $W = \text{span}(w_1, \dots, w_l)$.
Then $U \perp W \Leftrightarrow \langle u_i, w_j \rangle = 0$ for all meaningful i,j.

Proof ($\Rightarrow$) Suppose $U \perp W$. Then $\langle u_i, w_j \rangle = 0$
($\Leftarrow$) Suppose $\langle u_i, w_j \rangle = 0 \ \forall i,j$. Let $u \in U$, $w \in W$.
Then
$$u = c_1 u_1 + \cdots + c_k u_k$$
$$w = d_1 w_1 + \cdots + d_l w_l$$
$$\begin{aligned}\langle u, w \rangle &= \langle u, d_1 w_1 + \cdots + d_l w_l \rangle \\ &= \langle u, d_1 w_1 \rangle + \cdots + \langle u, d_l w_l \rangle \\ &= d_1 \langle u, w_1 \rangle + \cdots + d_l \langle u, w_l \rangle\end{aligned}$$
But then for each $\langle u, w_j \rangle$,
$$\begin{aligned}\langle u, w_j \rangle &= \langle c_1 u_1 + \cdots + c_k u_k, w_j \rangle \\ &= \langle c_1 u_1, w_j \rangle + \cdots + \langle c_k u_k, w_j \rangle \\ &= c_1 \langle u_1, w_j \rangle + \cdots + c_k \langle u_k, w_j \rangle \\ &= 0 + \cdots + 0 \\ &= 0\end{aligned}$$
So
$$\begin{aligned}\langle u, w \rangle &= 0 + \cdots + 0 \\ &= 0\end{aligned}$$

Thm For all matrix A, $\text{row } A \perp \text{nul } A$

Proof Let $A = \begin{bmatrix} - r_1^T - \\ \vdots \\ - r_m^T - \end{bmatrix}$

If $\vec{x} \in \text{nul } A$, $A\vec{x} = 0$

But $A\vec{x} = \begin{bmatrix} r_1^T \cdot \vec{x} \\ \vdots \\ r_m^T \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

So $\langle r_i, x \rangle = 0$

Let $\text{row } A = \text{span}(r_1, \dots, r_m)$
$\text{nul } A = \text{span}(n_1, \dots, n_p)$
Then $r_i \cdot n_j = 0 \ \forall i,j \Rightarrow \text{row } A \perp \text{nul } A$

Def Let V be a V.S with I.P $\langle \cdot, \cdot \rangle$ and W be supspace in V.
$$W^\perp = \{ v \in V \mid \forall w \in W, \langle v, w \rangle = 0 \}$$
$W^\perp$ is called the orthorgonal complement of W

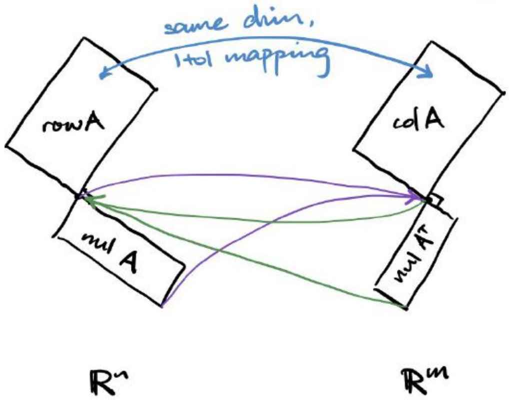
Thm $W^\perp$ is subspace in V

Proof Non-empty: $\langle \vec{0}, * \rangle = 0 \Rightarrow \vec{0} \in W^\perp$
Closure: let $u_1, u_2 \in W^\perp$, $c \in \mathbb{R}$
- $\langle u_1 + u_2, w \rangle = \langle u_1, w \rangle + \langle u_2, w \rangle = 0 + 0 = 0$
  $\Rightarrow u_1 + u_2 \in W^\perp$
- $\langle cu_1, w \rangle = c \langle u_1, w \rangle = 0$
  $\Rightarrow cu_1 \in W^\perp$

Thm $W \cap W^\perp = \{\vec{0}\}$

Proof [omitted]

Thm $\text{nul } A = (\text{row } A)^\perp$ and thus $\text{nul } A^T = (\text{row } A^T)^\perp = (\text{col } A)^\perp$



Thm Let $A_{m \times n}$, $A^T A$ invertible $\Leftrightarrow$ cols of A lin indep.

Proof ($\Leftarrow$) Suppose cols of A lin indep. WTS $A^T A \vec{x} = \vec{0} \Rightarrow \vec{x} = 0$
Well, $A^T A \vec{x} = \vec{0}$
$\Rightarrow A^T (A\vec{x}) = \vec{0}$
* $A\vec{x} \in \text{nul } A^T$
$\Rightarrow A\vec{x} \in (\text{col } A)^\perp$
* But $Ax \in \text{col } A$ ← lin comp of cols
$\Rightarrow \vec{x} = \vec{0}$