Lec 28

Orthonormal basis

<u>Notice</u> Let V be I.P.S. with some orthonormal basis $B = \{u_1, \dots, u_n\}$

Let $v, w \in V$, then

$$\left.\begin{aligned} v &= a_1 u_1 + \cdots + a_n u_n \\ w &= b_1 w_1 + \cdots + b_n w_n \end{aligned}\right\} \text{unique linear combo}$$

Then

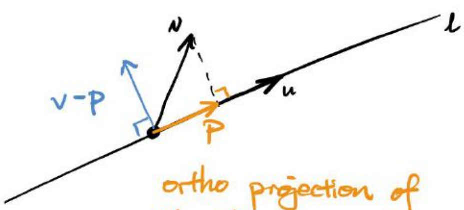
$$\langle v, w \rangle = a_1 b_1 + \cdots + a_n b_n$$

$$\langle v, v \rangle = \underbrace{a_1^2 + \cdots + a_n^2}_{\text{We get dot product like behaviours}} = \|v\|^2$$

We call $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ the <u>coordinate vec</u> for v w.r.t. B

Orthogonal Projections

Let $u \in \mathbb{R}^n$, $u \neq \vec{0}$, $\ell = \text{span}(u)$, $v \in \mathbb{R}^n$



ortho projection of v onto span(u)

$\Rightarrow$ p is the linear combo in span(u) such that $(v-p) \perp u$

Then $p = tu$ and $(v - tu) \cdot u = 0$

$$\Rightarrow u \cdot v - u \cdot (tu) = 0$$

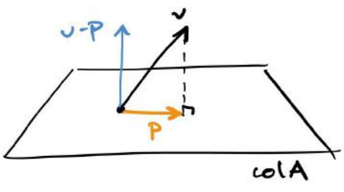
$$u \cdot v - t(u \cdot u) = 0$$

$$t = \frac{u \cdot v}{u \cdot u}$$

$$\boxed{\text{proj}_{\vec{u}} \vec{v} = \left(\frac{u \cdot v}{u \cdot u}\right) u}$$

$\rightarrow$ Want to be able to project onto any subspace

Let $A_{n \times p}$, project $v \in \mathbb{R}^p$ onto col A.



We must have ① $p \in \text{col } A$, ② $v - p \in (\text{col } A)^\perp$

Notice $\text{col } A = \{Ax \mid x \in \mathbb{R}^p\}$. Then

① $p = A\hat{x}$ f.s. $\hat{x} \in \mathbb{R}^p$

② Recall $(\text{col } A)^\perp = (\text{row } A^T)^\perp = \text{nul } A^T$

$\Rightarrow A^T(v - p) = \vec{0}$

Then WTFind $\hat{x}$ s.t. $A^T(v - A\hat{x}) = \vec{0}$

$$A^T v - A^T A\hat{x} = \vec{0}$$
$$A^T v = A^T A\hat{x}$$

WLOG suppose cols of A lin indep. (else we can throw out cols without changing col A)

Then $A^T A$ invertible.

$$\Rightarrow (A^T A)^{-1} A^T v = (A^T A)^{-1} A^T A \hat{x}$$
$$\hat{x} = (A^T A)^{-1} A^T v$$

So

$$\text{proj}_{\text{col} A}\, v = A\hat{x} = A(A^T A)^{-1} A^T v$$

Then, to project $v \in V$ into some subspace W, find matrix whose col space is W by taking basis of W and stacking horizontally. Then do the above work

<u>Thm</u> Orthogonal Decomposition Theorem.

If W is subspace of $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, then there are unique $w \in W$ and $u \in W^\perp$ s.t. $u + w = v$

<u>Proof</u> ($\exists$) Let $w = \text{proj}_W\, v$

$u = v - \text{proj}_W\, v$

Then $w \in W$, $u \in W^\perp$ by def and

$w + u = v$

(!) Suppose $w' \in W$, $u \in W^\perp$ s.t. $w' + u' = v$

Then $w + u = w' + u'$.

$$\underbrace{w - w'}_{\vec{x} \in W} = \underbrace{u' - u}_{\vec{x} \in W^\perp} \qquad \text{Let } \vec{x} = w - w'$$

Then $\vec{x} = \vec{0}$. So $w = w'$, $u = u'$