$\boxed{\text{Lec 29}}$

Recall orthogonal projection $\vec{v}$ onto span$(\vec{u})$ $\text{proj}_{\vec{u}} \vec{v} = \left(\frac{u \cdot v}{u \cdot u}\right) u$
or onto col(A) $\text{proj}_{\text{col}A} \vec{v} = A(A^TA)^{-1}A^T v$

Recall orth. decomp. thm.

Projection matrix

<u>Def</u> $A(A^TA)^{-1}A^T$ is a <u>projection matrix</u> as it sends sth to its projection in col A

<u>Thm</u> Let W be subspace for $\mathbb{R}^n$, $B^W = \{u_1, \dots, u_k\}$ be orth. basis for W, $B^{W^\perp} = \{u_{k+1}, \dots, u_n\}$ be orth. basis for $W^\perp$.
Then $B = B^W \cup B^{W^\perp} = \{u_1, \dots, u_n\}$ is orth. basis for $\mathbb{R}^n$

<u>Proof</u> WTS B orth., spans $\mathbb{R}^n$, is lin indep.

Let $i \neq j$, then $u_i \cdot u_j = 0$ because

$u_i, u_j \in B^W \Rightarrow u_i \cdot u_j = 0$
$u_i, u_j \in B^{W^\perp} \Rightarrow u_i \cdot u_j = 0$
$u_i \in B^W, u_j \in B^{W^\perp} \Rightarrow u_i \cdot u_j = 0$

B is set of mutually lin indep. non-zero vecs, so B is lin. indep.

Let $v \in \mathbb{R}^n$. Then $\exists! \, w \in W, u \in W^\perp$, $v = w + u$.
Then $v = c_1u_1 + \cdots + c_ku_k + c_{k+1}u_{k+1} + \cdots + c_nu_n$
$\in$ span(B)

$\square$

<u>Thm</u> Suppose $\{u_1, \cdots, u_k\}$ an orth. basis for subspace W in $\mathbb{R}^n$, then projection of $v \in \mathbb{R}^n$ onto W is

$$\text{proj}_W(v) = \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right) u_1 + \cdots + \left(\frac{u_k \cdot v}{u_k \cdot u_k}\right) u_k$$

<u>Proof</u> Know: $\text{proj}_W v \in W$
$\Rightarrow \text{proj}_W v = a_1u_1 + \cdots + a_ku_k$
Also: $\text{proj}_W v = p$ s.t. $w \cdot (v - p) = 0 \ \forall w \in W$.
$\Rightarrow v - p$ is orth. to W.
Then $\forall i = 1 \dots k$, $u_i \cdot (v - a_1u_1 + \cdots + a_ku_k) = 0$
$u_i \cdot v - u_i \cdot (a_1u_1 + \cdots + a_ku_k) = 0$
$u_i \cdot v - a_i(u_i \cdot u_i) = 0$
$a_i = \frac{u_i \cdot v}{u_i \cdot u_i}$

Orthogonal matrices

<u>Thm</u> If $Q_{m\times n}$ has orthonormal cols, then $Q^TQ = I_n$

<u>Proof</u>

Let $Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$

Then $Q^TQ = \begin{bmatrix} - & q_1 & - \\ & \vdots & \\ - & q_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = [q_i \cdot q_j]_{i,j}$

But $q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

So $Q^TQ = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

<u>Def</u> If $A_{n\times n}$ has orthonormal cols we say A is an <u>orthogonal matrix</u>

Viz. square matrix with orthonormal cols

Ex. $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

<u>Thm</u> $Q_{n\times n}$ is orth. matrix $\Leftrightarrow Q^TQ = I_n \Leftrightarrow Q^T = Q^{-1}$

<u>Def</u> A matrix P is a <u>permutation matrix</u> if its cols are standard basis vec for $\mathbb{R}^n$ in any order

Ex. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

<u>Note</u> any permutation matrix is orthogonal

<u>Thm</u> Let $Q_{n\times n}$, $\vec{x}, \vec{y} \in \mathbb{R}^n$, TFAE

1. Q is orth. matrix
2. $\|Q\vec{x}\| = \|\vec{x}\|$ ← preserves length
3. $(Q\vec{x})\cdot(Q\vec{y}) = \vec{x}\cdot\vec{y}$ ← preserves dot prod

<u>Proof</u> ① ⇒ ③ $(Qx)\cdot(Qy) = (Qx)^T(Qy)$

$= x^T \cancel{(Q^TQ)}\, y$

$= x \cdot y$

③ → ② Suppose $(Qx)\cdot(Qy) = x\cdot y$. Let $x = y$

Then $\|Qx\| = \sqrt{(Qx)\cdot(Qx)}$

$= \sqrt{x\cdot x}$

$= \|x\|$

③ → ① [omit]