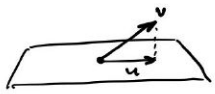
Lec 31

Least Square Approximation

Consider: want to solve $Ax = b$. Maybe there's no solution ... but we want some approximate solution. (aka inconsistent)

Want x^* s.t. Ax^* is as close to b as possible.

<u>Thm</u> Best approx theorem: if W subspace in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, then $u \in W$ closest to v is $\text{proj}_W v$.



Then, to find x^*, we solve $Ax^* = \text{proj}_{\text{col}(A)} b$. Viz. minimise $b - Ax^*$.

Meanwhile $b - Ax^* \in (\text{col} A)^\perp \Rightarrow b - Ax^* \in \text{nul}(A^T)$.

$$\Rightarrow \text{want } A^T(b - Ax^*) = \vec{0}$$
$$A^Tb - A^TAx^* = \vec{0}$$
$$A^Tb = A^TAx^*$$

← the normal equation for $Ax = b$ always has solution.

or, ... think about error $\text{err} = b - Ax = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{bmatrix}$

$\|\text{err}\| = \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_m^2}$ ← WT minimise

Regression application

Suppose we have (x_i, y_i) and expect $y_i = mx_i + b$

x	y
1	1
3	2
2	3
3	1

Want:

$$m + b = 1$$
$$3m + b = 2$$
$$2m + b = 3$$
$$3m + b = 1$$

$$\rightsquigarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}}_{A} \begin{bmatrix} m \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}}_{b}$$

← inconsistent, but still want a least square solution

$$A^TA = \begin{bmatrix} 23 & 9 \\ 9 & 4 \end{bmatrix} \qquad A^Tb = \begin{bmatrix} 16 \\ 7 \end{bmatrix}$$

→ Normal eq: $A^Tb = A^TAx^*$

$$\begin{bmatrix} 16 \\ 7 \end{bmatrix} = \begin{bmatrix} 23 & 9 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} m^* \\ b^* \end{bmatrix}$$

$$\cdots \Rightarrow \begin{bmatrix} m^* \\ b^* \end{bmatrix} = \begin{bmatrix} 1/11 \\ 17/11 \end{bmatrix}$$

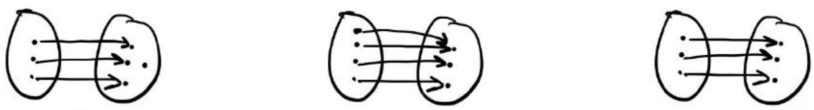
Thm Let $A_{m\times n}$, $b \in \mathbb{R}^m$, then there's at least a least square sol to $Ax=b$.
Moreover:

1. x^* is L.S.S. $\Leftrightarrow$ $A^TAx^* = A^Tb$
2. If cols of A lin. indep, then x^* is unique.

Def If A has lin indep cols
$A^+ = (A^TA)^{-1}A^T$ is the pseudoinverse of A.

$\Rightarrow$ If A invertible $\Rightarrow$ $Ax=b$ has unique sol $x = A^{-1}b$.
A lin indep cols $\Rightarrow$ $Ax=b$ has least square sol $x^* = A^+b$
But if A invertible $A^+ = (A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}$
So $x = x^*$ in this case.

L.T. as functions



one-to-one / injective / monomorphism

onto / surjective / epimorphism

one-to-one and onto / bijective / isomorphism

$T: V \to W$
- W — codomain
- V — domain

$\text{im}(T) = \{T(v) \mid v \in V\} \subseteq W$ — image of T

$\ker(T) = \{v \in V \mid T(v) = \vec{0}_W\}$ — kernel of T

Then
- T injective $\Leftrightarrow$ $\ker T = \vec{0}_V$
- T surjective $\Leftrightarrow$ $\text{im}\, T = W$