Lec 32

Coordinate w.r.t. basis

Def Let $B = \{b_1, \dots, b_n\}$ be basis for V.S. V.
Notice all $v \in V$ is some unique lin combo $v = c_1 b_1, \dots, c_n b_n$.
We say the coords of v w.r.t. B are $c_1, \dots, c_n$
And the coord vec of v w.r.t. B is

$$[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \leftarrow \text{Notice this is in } \mathbb{R}^n$$

Ex. Consider:

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad v = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

$$\cdots \Rightarrow \left[\begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} \right]_B = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}$$

Ex. Consider

$$B = \{1, x, x^2\} \text{ for } P_2 . \qquad v = 1 + x^2$$

$$\cdots \Rightarrow [1+x^2]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \{1, 1+x, 1+x+x^2\}$$

$$\cdots \Rightarrow [1+x^2]_C = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Lemma Operating with coord vec corresponds to ops in old coord sys

1. $[u+v]_B = [u]_B + [v]_B$
2. $[cv]_B = c[v]_B$

[proof omitted]

Coro If V is v.s with basis B, then

$$C : V \to \mathbb{R}^n$$
$$v \in V \mapsto [v]_B$$

is an invertible L.T. viz. C is bijective

Notice Let $B = \{b_1, \dots, b_n\}$ be basis. Then

$$[b_i]_B = \begin{array}{c} \\ i\text{th} \rightarrow \\ \\ \end{array} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e_i$$

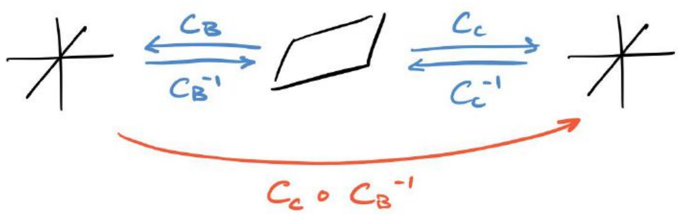
* Change of basis

Let v.s. V, basis $B = \{u_1, \dots, u_n\}$, basis $C = \{v_1, \dots, v_n\}$, $x \in V$.

Then $x = c_1u_1 + \cdots + c_nu_n$
$\quad\;= d_1v_1 + \cdots + d_nv_n$ viz. $[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ $[x]_C = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$

But meanwhile:

$$\begin{array}{ccccc} & \text{some L.T.} & & \text{some L.T.} & \\ [x]_B & \longleftrightarrow & x & \longleftrightarrow & [x]_C \\ \mathbb{R}^n & & V & & \mathbb{R}^n \end{array}$$



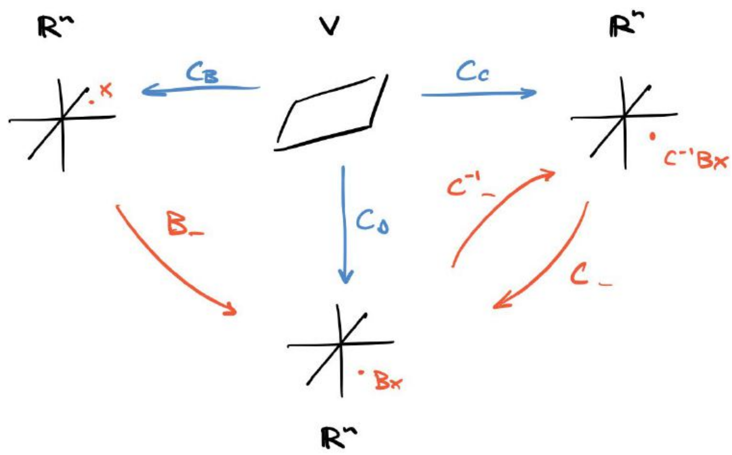
Consider $B = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$ Recall $Be_i = u_i$

But $u_i = [u_i]_{\mathcal{S}}$ ← standard basis $\quad e_i = [u_i]_B$

$\Rightarrow \quad B[u_i]_B = [u_i]_{\mathcal{S}}$

But then
$$\begin{aligned} B[x]_B &= B[c_1u_1 + \cdots + c_nu_n]_B \\ &= B(c_1[u_1]_B + \cdots + c_n[u_n]_B) \\ &= c_1(B[u_1]_B) + \cdots + c_n(B[u_n]_B) \\ &= c_1[u_1]_{\mathcal{S}} + \cdots + c_n[u_n]_{\mathcal{S}} \\ &= [c_1u_1 + \cdots + c_nu_n]_{\mathcal{S}} \\ &= [x]_{\mathcal{S}} \end{aligned}$$

So



$B = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$

$C = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

<u>Useful</u> $\boxed{[\,C \mid B\,] \rightsquigarrow \cdots \rightsquigarrow [\,I_n \mid C^{-1}B\,]}$

Ex.

$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$[\,C \mid B\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$\cdots \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$

$\qquad\qquad I_3 \qquad\quad C^{-1}B$

<u>Def</u> We call this $C^{-1}B$ the change of basis matrix. Write:

$$P_{C \leftarrow B} = C^{-1}B$$

And correspondingly $[x]_B \mapsto [x]_C$ is linear map

viz. $(P_{C \leftarrow B})\,[x]_B = [x]_C$