$\boxed{\text{Lec 33}}$

Recap

Recall for basis $B = \{b_1, \dots, b_n\}$ in V there's unique encoding for $v \in V$ in terms of basis in B $[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$.

And we can have bijective L.T. $C_B : V \to \mathbb{R}^n$ that sends $v \mapsto [v]_B$

In special case where $V = \mathbb{R}^n$, we have std basis $\mathcal{S} = \{e_1, \dots, e_n\}$.
And so all $\forall x \in \mathbb{R}^n$, $[x]_\mathcal{S} = x$

Consider

$B = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{bmatrix}$. Then $Be_i = b_i$ $\cdots \Rightarrow$ $B[x]_B = [x]_\mathcal{S}$, $B^{-1}[x]_\mathcal{S} = [x]_B$ for any $x \in \mathbb{R}^n$

Picking helpful coords

Consider generic v.s. V and $\mathbb{R}^n$ with some basis $B = \{b_1, \dots, b_n\}$
$C = \{c_1, \dots, c_n\}$

We want $P_{C \leftarrow B} [b_i]_B = [b_i]_C$.

Then $P_{C \leftarrow B} = \begin{bmatrix} | & & | \\ [b_1]_C & \cdots & [b_n]_C \\ | & & | \end{bmatrix}$

Ex. $B = \{1, 1+x, 1+x^2\}$
$C = \{1+x, 1+x+x^2, x+x^2\}$

Well $b_1 = 1 = (0)(1+x) + (1)(1+x+x^2) + (-1)(x+x^2)$
$b_2 = 1+x = (1)(1+x) + (0)(1+x+x^2) + (0)(x+x^2)$
$b_3 = 1+x^2 = (-1)(1+x) + (2)(1+x+x^2) + (-1)(x+x^2)$

So $[b_1]_C = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $[b_2]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $[b_3]_C = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

$$P_{C \leftarrow B} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

<u>Thm</u>
1. $P_{C \leftarrow B} [x]_B = [x]_C$ $\forall x \in B$
2. $P_{C \leftarrow B}$ is unique
3. $P_{C \leftarrow B}$ invertible and $(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}$