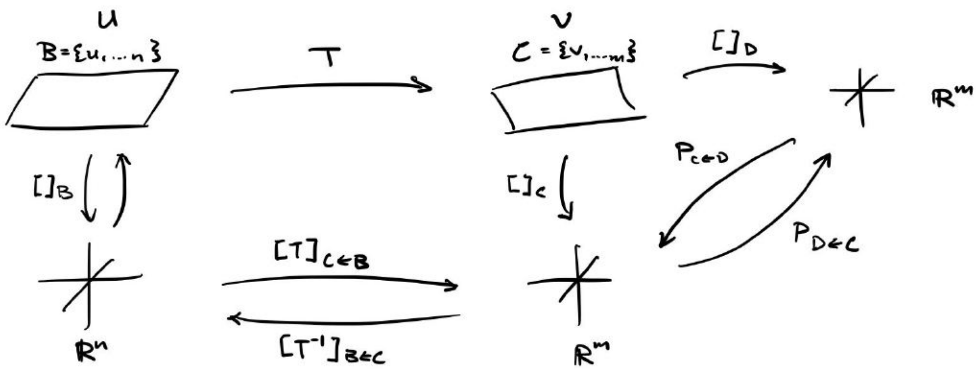
Lec 35

Alt notation for C_B : $[\cdot]_B$

More complicated V.S. mappings



Notice: $[T^{-1}]_{B\leftarrow D} = [T^{-1}]_{B\leftarrow C} P_{C\leftarrow D}$

Ex. $T : U \to V$ via $T(\vec{x}) = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} \vec{x}$

$U = \mathbb{R}^2$ $\quad B = C = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$V = \mathbb{R}^2$

$\left[T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right]_B = \left[\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right]_B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\qquad \left[T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right]_B = \left[\begin{bmatrix} -4 \\ 4 \end{bmatrix}\right]_B$

So $[T]_{B\leftarrow B} = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$ ← Nice diagonal matrix because we chose basis in the eigenspace of $\begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$

But also $[T]_{S\leftarrow S} = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$

$P_{S\leftarrow B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$P_{B\leftarrow S} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

So $[T]_{S\leftarrow S} = P_{S\leftarrow B} [T]_{B\leftarrow B} P_{B\leftarrow S} = (P_{B\leftarrow S})^{-1} [T]_{B\leftarrow B} P_{B\leftarrow S}$

Indeed $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$

Does that mean $[T]_{S\leftarrow S}$ and $[T]_{B\leftarrow B}$ are somehow connected?

Def We say A and B are similar if $\exists P$ s.t. $P^{-1}AP = B$
Viz. they represent same L.T. for basis that could be different.
We write $A \sim B$

Thm Let A, B, C be $n \times n$. Then:

1. $A \sim A$
2. $A \sim B \Leftrightarrow B \sim A$
3. $A \sim B \wedge B \sim C \Rightarrow A \sim C$

Thm Let A, B be $n \times n$ with $A \sim B$. These are true:

1. $\det A = \det B$
2. $\exists A^{-1} \Leftrightarrow \exists B^{-1}$
3. $\operatorname{rank} A = \operatorname{rank} B$
4. A, B have same charastic polynomial $\det(A - \lambda I)$
5. A, B have same evecs

Proof for 4

Well
$$\begin{aligned}
\det(B - \lambda I) &= \det(P^{-1}AP - \lambda P^{-1}IP) \\
&= \det(P^{-1}(AP - \lambda IP)) \\
&= \det(P^{-1}(A - \lambda I)P) \\
&= \det(P^{-1}) \det(A - \lambda I) \det(P) \\
&= \det(P^{-1}P) \det(A - \lambda I) \\
&= \det(A - \lambda I)
\end{aligned}$$