$\boxed{\text{Lec 36}}$

Diagonisability

<u>Def</u> $A_{n\times n}$ is diagonalisable if $\exists$ diagonal mactrix D, $\exists P$, $\exists P^{-1}$, s.t.
$P^{-1}AP = D$

Viz. it's similar to some diagonal matrix

<u>Thm</u> $A_{n\times n}$ diagonalisable $\Leftrightarrow$ A has n lin indep e-vecs
In that case
$P^{-1}AP = D \Rightarrow$ cols of P are e-vals and entries of D are corresponding e-vals

<u>Proof</u>

Suppose $P^{-1}AP = D$. Bind $\begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix} = P$, $\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} = D$

$$\Rightarrow AP = PD$$

$$A \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

$$\begin{bmatrix} | & & | \\ Ap_1 & \cdots & Ap_n \\ | & & | \end{bmatrix} = \sum \begin{bmatrix} | \\ p_i \\ | \end{bmatrix} [0 \cdots 0 \, d_i \, 0 \cdots 0]$$

$$= \sum \begin{bmatrix} \vec{0} & \cdots & \vec{0} & d_i \begin{bmatrix} | \\ p_i \\ | \end{bmatrix} & \vec{0} & \cdots & \vec{0} \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ d_1 p_1 & \cdots & d_n p_n \\ | & & | \end{bmatrix}$$

So $Ap_i = d_i p_i$ for $i \in 1..n$

So 1. cols of P are e-vecs of A
2. entries in D are corresponding e-vals

So cols of P form eigenbasis for $\mathbb{R}^n$

Now suppose $\{p_{1..n}\}$ is e-basis for A. Viz $Ap_i = \lambda_i p_i$

Let $P = \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Notice the i^{th} col of $P^{-1}AP$ is $P^{-1}APe_i$

$$\begin{aligned} &= P^{-1}Ap_i \\ &= P^{-1}\lambda_i p_i \\ &= \lambda_i P^{-1}p_i \quad \} \; Pe_i = p_i \Rightarrow e_i = P^{-1}p_i \\ &= \lambda_i e_i \end{aligned}$$

So $P^{-1}AP = \begin{bmatrix} \cdots & \lambda_i e_i & \cdots \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= D$$

<u>Obsticles</u>

- If geo multiplicity < algebraic multiplicity, then we don't get diagonalisation
- Complex e-vals ?

<u>Thm</u> If $A_{n \times n}$ with $\lambda_{1..k}$ and distinct e-vecs $B_i = \{v_{i1}, \dots, v_{in_i}\}$ as basis for E_{λ_i}, then $B = \bigcup B_i$ is a lin indep set.

<u>Proof</u>

$B = \{v_{11}, \dots, v_{1n_1}, \dots, v_{k1}, \dots, v_{kn_k}\}$

Suppose $\underbrace{c_{11}v_{11} + \cdots + c_{1n_1}v_{1n_1}}_{u_1 \in E_{\lambda_1}} + \cdots + \underbrace{c_{k1}v_{k1} + \cdots + c_{kn_k}v_{kn_k}}_{u_k \in E_{\lambda_k}} = \vec{0}$

$$u_1 + \cdots + u_k = \vec{0} \quad \text{for } u_i \in E_{\lambda_i}$$

If any $v_i \neq 0$, we could solve it in terms of others, but we can't as e-vecs from different e-vals are lin indep.

So $c_{11}, \dots, c_{in_i} = 0$, and B is lin indep.

<u>Thm</u> If $A_{n \times n}$ with distinct e-vecs $\lambda_{1..k}$, TFAE

1. A is diagonalisable
2. $B = \bigcup B_i$ has n vecs
3. $\forall \lambda_i$, its geo multiplicity = algebraic multiplicity

Ex. $A = \begin{bmatrix} 1/2 & 3/4 & -5 \\ 5 & 1/2 & -10 \\ 0 & 0 & -2 \end{bmatrix}$ with $\lambda_1 = 3$, $\lambda_2 = -2$

Double root

$B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ $\quad B_2 = \underbrace{\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\dim = 2}$

So $\forall \lambda_i$, its geo multiplicity = algebraic multiplicity

So A is diagonalisable

We can take $P = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Note this is not the unique way