$\boxed{\text{Lec 37}}$

Recall: Diagonalisation as special case of similarity

Orthogonal diagonalisation of real symmetric matrices
- orthonormal basis of e-vecs

<u>Def</u> $A_{n\times n}$ is <u>orthogonally diagonalisable</u> if $\exists$ orth. matrix Q s.t.

$$Q^T A Q = D = Q^{-1} A Q$$

recall Q orth $\Rightarrow Q^T = Q^{-1}$

<u>Def</u> A is <u>symmetric</u> if $A = A^T$

<u>Thm</u> If A is orthogonally diagonalisable, then A is symmetric.

<u>Proof</u>

$$\begin{aligned} Q^T A Q &= Q^{-1} A Q = D \\ A &= QDQ^{-1} = QDQ^T \\ A^T &= (Q^T)^T D^T Q^T \\ &= QDQ^T \\ &= A \end{aligned}$$

<u>Ex.</u>

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{with } \lambda_1 = 4,\ \lambda_2 = 1$$

$$E_4 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad E_1 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\downarrow$ Gram-Schmidt

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\}$$

normalise

$$Q^T A Q^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1/2}{\sqrt{3/2}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3/2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1/2}{\sqrt{3/2}} \end{bmatrix}$$

<u>Thm</u> Every real symmetric matrix B orthogonally diagonalisable

Complex e-val — related to rotation.

Ex $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Bind $\begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\det(R - \lambda I) = 0$$
$$(c-\lambda)^2 + s^2 = 0$$
$$\lambda^2 - 2c\lambda + (c^2+s^2) = 0$$
$$\lambda^2 - 2c\lambda + 1 = 0$$
$$\lambda = c \pm \sqrt{c^2-1}$$
$$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$$
$$= \cos\theta \pm i\sin\theta$$

But then $A(\overbrace{v+iw}^{\text{complex vec}}) = (\alpha - i\beta)(v+iw)$

$Av + iAw = (\alpha v + \beta w) + i(-\beta v + \alpha w)$

$\Rightarrow Av = \alpha v + \beta w$,

$Aw = -\beta v + \alpha w$

Claim : $B = \{v, w\}$ is a basis and

$[Av]_B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$[Aw]_B = \begin{bmatrix} -\beta \\ \alpha \end{bmatrix}$

Then $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} = \frac{1}{\alpha^2+\beta^2} \begin{bmatrix} \frac{\alpha}{\alpha^2+\beta^2} & -\frac{\beta}{\alpha^2+\beta^2} \\ \frac{\beta}{\alpha^2+\beta^2} & \frac{\alpha}{\alpha^2+\beta^2} \end{bmatrix}$

Set $\cos\theta = \frac{\alpha}{\alpha^2+\beta^2}$, $\sin\theta = \frac{\beta}{\alpha^2+\beta^2}$

So this is a rotation matrix.