$\boxed{\text{Lec 7}}$ Vector valued functions and space curves

* Vector valued func : $\vec{r} : \mathbb{R} \to \mathbb{R}^d$

* space curve is <u>its</u> graph

Ex. helix in $\mathbb{R}^3$ — $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

Calc can help find:

- Arc length of space curve
- Tangent line of space curve
- Curvature

Limit & derivative for space curve

$$\lim_{t \to t_0} = \left\langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right\rangle$$

(if all limits exist)
← generalizes to higher dims
← imagine approaching each coord.

Derivative

$$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - r(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle$$

← Notice we get some tangent vector

also the instantaneous velocity

Tangent line.

Example. find tangent vec and line for $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $(2,4,8)$.

So $t = 2$

tangent vector

$r'(t) = \langle 1, 2t, 3t^2 \rangle$ $\quad r'(2) = \langle 1, 4, 12 \rangle$

$\ell(t) = \langle 2,4,8 \rangle + t\langle 1,4,12 \rangle$

Principle unit tangent vector

Let $\vec{r}(t)$ be space curve

Def: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ when $\vec{r}\,'(t)$ exists and non-zero

Basically we get rid of the parametrisation.
Just get unit vec for direction info

Integrals of space curves

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

think of t as time and r(t) is velocity.
then we integrate to find position.

Ex. velocity $\vec{v}(t) = \langle t, t^2, t^3 \rangle$
start position $(0,1,0)$ at rest
find position at $t=2$.

$$\int \vec{v}(t)\,dt = \left\langle \frac{t^2}{2}, \frac{t^3}{3}, \frac{t^4}{4} \right\rangle + C$$

$$\vec{r}(0) = \langle 0,1,0 \rangle \Rightarrow C = \langle 0,1,0 \rangle$$

$$\Rightarrow \vec{r}(t) = \left\langle \frac{t^2}{2}, \frac{t^3}{3}, \frac{t^4}{4} \right\rangle + \langle 0,1,0 \rangle$$

$$\vec{r}(2) = \left\langle 2, \frac{11}{3}, 4 \right\rangle$$

So $\left(2, \frac{11}{3}, 4\right)$.

Arc length

$$s = \int_a^b \|\vec{r}\,'(t)\|\,dt.$$