Lec 8

We want to try standardise parameter so that "speed of particle" doesn't matter.

Reparameterisation by arc length

→ by 1 unit of distance (arc length) per unit of time

Def: Let $\vec{r}(t)$ be space curve for $a \le t \le b$. Then arc length function:

$$s(t) = \int_a^t \|\vec{r}\,'(t)\| \, du \quad \text{for } t \in [a,b]$$

Consider $\vec{r}(t) = \langle t^2, \underset{?}{2t}, 1 \rangle$ for $0 \le t < \infty$

$\vec{r}\,'(t) = \langle 2t, \underset{?}{2}, 0 \rangle$

⚠ this does not follow. The original curve might have been $\vec{r}(t) = \langle t^2, 2t^2, 1 \rangle$; in that case this makes sense

$$\Rightarrow s(t) = \int_0^t \sqrt{4u^2 + 16u^2} \, du = \int_0^t \sqrt{20}\, u \, du = \sqrt{20}\, t^2$$

Then we solve $s = s(t)$ for t

$$s = s(t) = \sqrt{20}\, t^2$$

$$t^2 = \frac{s}{\sqrt{5}}$$

$$t = \sqrt{\frac{s}{\sqrt{5}}}$$

Now $\vec{r}(s)$ moves at standard speed

$$\vec{r}(t) = \langle t^2, 2t, 1 \rangle$$

$$\Rightarrow \vec{r}(s) = \left\langle \left(\sqrt{\frac{s}{\sqrt{5}}}\right)^2, 2\left(\sqrt{\frac{s}{\sqrt{5}}}\right), 1 \right\rangle$$

Principle unit normal vector

Def: $$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

Curvature

Take smooth curve $\vec{r}(t)$

Def: curvature $K = \dfrac{d\,\|T(s)\|}{ds}$ where $T(s)$ is principle tangent vector

↙ magnitude of change in direction

↖ per change in arc length

This is kind of hard to calculate ... we need to find find $s(t)$ and $s'(t)$ and $T(s)$ ...

Hmm there's a shortcut in $\mathbb{R}^3$
↓
Theorem: if $\vec{r}(t)$ in $\mathbb{R}^3$, then

$$K(t) = \frac{\|T'(t)\|}{\|r'(t)\|}$$

and

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

end

↑ Not most straightforward

Proof

By def, $K = \left\|\dfrac{dT}{ds}\right\| = \left\|\dfrac{dT/dt}{ds/dt}\right\|$.

- $\left\|\dfrac{dT}{dt}\right\| = \|T'(t)\|$
- $\left\|\dfrac{ds}{dt}\right\| = \|r'(t)\|$

$\Rightarrow \dfrac{\|T'(t)\|}{\|r'(t)\|}$

Theorem : $T \cdot T' = 0$
- $T \cdot T = 1$; $(T \cdot T)' = 1' \Rightarrow T' \cdot T + T \cdot T' = 0 \Rightarrow T \cdot T' = 0$

※ CURRENT

Theorem: $\vec{r}(t)$, $\vec{q}(t)$ be space curves, then

$$\frac{d}{dt}(\vec{r} \cdot \vec{q}) = \vec{r}\,' \cdot \vec{q} + \vec{r} \cdot \vec{q}\,' \quad \text{and}$$

$$\frac{d}{dt}(\vec{r} \times \vec{q}) = \vec{r}\,' \times \vec{q} + \vec{r} \times \vec{q}\,'$$

→ Now WTS $\|T'(t)\| = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^2}$

Recall $T(t) = \dfrac{r'(t)}{\|r'(t)\|}$ and $\|r'(t)\| = \dfrac{ds}{dt}$

So $r'(t) = \|r'(t)\|\, T(t) = \dfrac{ds}{dt} T(t)$

$r''(t) = \dfrac{ds}{dt} T'(t) + \dfrac{d^2s}{dt^2} T(t)$

Then $r'(t) \times r''(t) = \dfrac{ds}{dt} T \times \left(\dfrac{ds}{dt} T' + \dfrac{d^2s}{dt^2} T\right)$

$= \left(\dfrac{ds}{dt}\right)^2 T \times T' + \dfrac{ds}{dt}\left(\dfrac{d^2s}{dt^2}\right) \cancel{T \times T}^{\,0}$

$= \left(\dfrac{ds}{dt}\right)^2 T \times T'$

Then $\|r' \times r''\| = \left(\dfrac{ds}{dt}\right)^2 \|(T \times T')\|$

$= \left(\dfrac{ds}{dt}\right)^2 \cancel{\|T\|}^{\,1} \|T'\| \cancel{\sin\theta}^{\,1}$

$= \|r'(t)\|^2 \|T'\|$