Lec 12 Linear Approximation and Differentials

Definitions

Def: if $f(x,y)$ has both partial derivative defined at (a,b), the linear approx is the plane:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Def: say that f is differentiable at (a,b) if the linear approximation is the tangent plane

i.e. differentiable means tangent line exist in every direction and they live in the same plane (would be the tangent plane)

formally, $f(x,y)$ is differentiable at (a,b) if:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\text{dist}((x,y),(a,b))} = 0$$

Theorem: If f_x and f_y exist and are continuous at (a,b) and f is differentiable at (a,b)

Ex: $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

continuous, but not differentiable

Notice both f_x and f_y exist at $(0,0)$, namely $\nabla f = (0,0)$
However, f_x not continuous along $y=0$. Simili for f_y.
So linear approx exists but tangent plane doesn't

But consider the $x=y$ direction, $f(x) = \frac{x^2}{\sqrt{2x^2}} = \frac{x^2}{\sqrt{2}|x|} = \frac{|x|}{\sqrt{2}}$ no derivative at $(0,0)$

Differentiables

If $f(x,y)$ is differentiable at (a,b), then $L(x,y) \approx f(x,y)$

Ex. approximate $z = 2x^2 - y^2$ at $(1.1, 0.9)$ using linear approx. at $(1,1)$.

$\frac{\partial z}{\partial x} = 4x \quad \frac{\partial z}{\partial y} = -2y$. $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= 1 + 4(x-1) - 2(y-1)$

$L(1.1, 0.9) = 1 + 0.4 + 0.2 = 1.6$

If $z = f(x,y)$ differentiable, its differential at (a,b) is :

$$dz = f_x(a,b)\,dx + f_y(a,b)\,dy$$

differential for $y = f(x)$ is $dy = f'(a)\,dx$ at $x = a$.

* Differentiability $dz \approx \Delta z$

$dz = 4x\,dx - 2y\,dy$.

At $dx = 0.1$, $dy = -0.1$, $dz = 0.6 \Rightarrow \Delta z \approx 0.6$.

$f(1.1, 0.9) \approx f(1,1) + dz = 1 + 0.6 = 1.6$

Ex: appromating measurement error.

Suppose sides of rectange are 2, 3 with ± 0.2 in each dir. How accurate is area?

Let dx be error in x and dy be error in y. Then error in $z = f(x,y)$ is

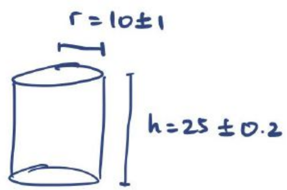
$$\pm\left(|f_x(a,b)\,dx| + |f_y(a,b)\,dy|\right)$$

$$= \pm\left(|3 \cdot 0.2| + |2 \cdot 0.2|\right) = \pm 1.0$$

Intuition :

$f(x)$ at some y $\qquad$ $f(y)$ at some x

Ex :



$$V = \pi r^2 h$$

$$dV = 2\pi r h\, dr + \pi r^2\, dh$$

$$\text{Error} = \pm \left(|2\pi \cdot 10 \cdot 25 \cdot 0.1| + |\pi \cdot 10^2 \cdot 0.2| \right)$$

$$= \pm 70\pi$$