Lec 14

Directional derivatives & computing using gradient

$f(x,y)$

$\frac{\partial f}{\partial x}$ ← Directional deri. in x dir $\vec{u} = \langle 1, 0 \rangle$

Def: Given unit vector $\vec{u} = \langle \cos\theta, \sin\theta \rangle$, (A unit vector) directional deri. of $f(x,y)$ is

$$D_{\vec{u}} f \; \lim_{h \to 0} \frac{f(x + h\cos\theta,\, y + h\sin\theta) - f(x,y)}{h}$$

Thm: If $f(x,y)$ differentiable at (a,b), then

$$D_{\vec{u}} f = \nabla f(a,b) \cdot \vec{u}$$

Fml: $D_{\vec{u}} f(a,b)$ in dir of $\langle c, d \rangle$ is

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \frac{\langle c, d \rangle}{\| \langle c, d \rangle \|}$$ ← Normalised vector

$f(x,y) = x^2 - 2xy + 3y^2$

Want $D_{\vec{u}} f(1,2)$ in $\langle -3, 4 \rangle$.

$\vec{u} = \frac{\langle -3, 4 \rangle}{\| \langle -3, 4 \rangle \|} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

$\nabla f = \langle 2x - 2y, -2x + 6y \rangle$

$D_{\vec{u}} f = \langle -2, 10 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle$

$= \frac{46}{5}$

* Note these things apply to higher dims

Proof: Let $g(h) = f(a + h\cos\theta, b + h\sin\theta)$

By def, $D_{\vec{u}} f \; \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$

Meanwhile, $g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$ where $x = a + h\cos\theta$, $y = b + h\sin\theta$

$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$

* Corollary: If $\nabla f(a,b) = \vec{0}$, then all dir. deri. at (a,b) are 0.

* If f differentiable where $\nabla f = \vec{0}$, it's a critical point

Direction of steepest increase

* ∇f points in dir of steepest increase (when $\nabla f \neq 0$)

Proof: WTS $\max_{\vec{u}} D_{\vec{u}} f$ attained when $\vec{u} = \frac{\nabla f}{\|\nabla f\|}$

Well ... $D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta$

$= \|\nabla f\| \cos\theta$.

This is maximised when $\theta = 0$ i.e. ∇f and $\vec{u}$ in same dir

Normal to level curve

Thm: for differentiable $f(x_1, \dots, x_d)$ at $(a_1, \dots, a_d)$ on level set $k = f(a_1, \dots, a_d)$, $\nabla f(a_1, \dots, a_d)$ is normal to the level set.

Ex. Consider $2x^2 + y^2 + 3z^2 = 6$ at $(1,1,1)$

Let $w = 2x^2 + y^2 + 3z^2$

$\nabla w = \langle 4x, 2y, 6z \rangle$

$\nabla w(1,1,1) = \langle 4, 2, 6 \rangle$ ← normal to level curve!

Tangent plane: $4(x-1) + 2(y-1) + 6(z-1) = 0$

Proof (for $d = 2$). Let $z = f(x,y)$. Get level curve $k = f(x,y)$

Parametrise level curve as $a = \langle x(t), y(t) \rangle$

WTS $\nabla f \cdot \langle x'(t), y'(t) \rangle = 0$ ← sth dot tangent vec $= 0$ is orthogonal to it

$\nabla f \cdot \langle x'(t), y'(t) \rangle = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$= \frac{\partial f}{\partial t} = 0$ ← because t isn't changing