Lec 15

Local extrema

Recall 2D strat:

1. Find where $f'(x) = 0$
2. Check $f''(x)$ to see if it's min or max

3D:

Def: Local maximum for $f(x,y)$ is (x_0, y_0) when there exists $\delta > 0$ s.t. $\text{dist}((x,y),(x_0,y_0)) < \delta \Rightarrow f(x_0,y_0) \geq f(x,y)$

If f differentiable, then a critical point (x_0, y_0) is when $\nabla f(x_0,y_0) = \vec{0}$

If $\nabla f(x_0,y_0)$ is undefined, then (x_0,y_0) is critical point too.

Thm: If f differentiable and has local max or min at (x_0,y_0) then (x_0,y_0) is a crit point.

Types of crit points

- Local max
- Local min
- Saddle point

decide by

Second derivative test

Thm: for a C^2 function ($f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{yx}$ exist and continuous)

Let $D = \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx}$ at crit point (x_0, y_0)

If $D(x_0,y_0) < 0$, then saddle point
If $D(x_0,y_0) = 0$, inconclusive
If $D(x_0,y_0) > 0$, then
  If f_{xx} or $f_{yy} > 0$, then local min
  If f_{xx} or $f_{yy} < 0$, then local max

Clairaut theorem

If $f(x,y)$ is C^2 then $f_{xy} = f_{yx}$

Sketch proof

First recall proof for $y = f(x)$:

If f is C^2!

$\rightarrow 0$ since crit point

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

$\leftarrow$ so sign of this determines

$$f(x) = f(x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 \quad \text{near } x_0$$

In case of $z = f(x,y)$, quadratic approx is

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$+ f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2$$

At crit point:

$$f(x,y) \approx f(x_0,y_0) + f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2$$

If $D \neq 0$, $f(x,y) \approx ax^2 + 2bxy + cy^2$, which is either <u>elliptic or hyperbolic paraboloid</u>

Which of — and their dir of opening tells us which one.

Ex: find & classify crit points of $x^3 + 4xy - 2y^2$

$f_x = 3x^2 + 4y \underline{= 0}$ $\qquad f_y = 4y - 4y \underline{= 0}$ $\qquad$ set

$\cdots \Rightarrow (x,y) = (0,0)$ or $(x,y) = (-8/3, -16/3)$

Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 4 \\ 4 & -2 \end{bmatrix}, \quad D = -12x - 16$$

$D(0,0) = -16 < 0 \rightarrow$ saddle point

$D(-8/3, -16/3) = * > 0$, and $f_{yy}(0,0) = -2 < 0 \rightarrow$ local max.