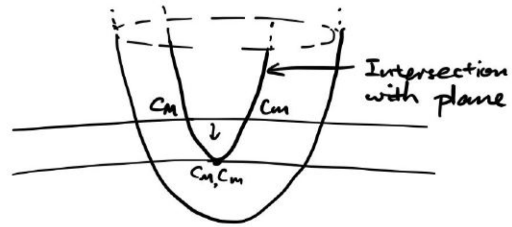
Lec 16

Constrained optimisation

i.e. finding min / max of f subjected to $g = k$.



Ex. dist from $(1,-1)$ to $2x - y = 6$

Ex. find min of $f(x,y) = (x-1)^2 + (y+1)^2$ subjected to $2x - y = 6$

* Lagrange multipliers — optimising f subjected to $g = k$

↳ find max / min c_M, c_m s.t. $c_M = f$, $c_m = f$ intersects $g = k$

<u>Thm</u>: then the level sets $f = c_M$ and $f = c_m$ are tangent to $g = k$

Now recall ∇f is normal to level set $f = c$ $\Rightarrow$ $f = c$ tangent to $g = b$ $\Rightarrow$ at that point ∇f parallel to ∇g. So $\exists \lambda \in \mathbb{R}$, $\nabla f = \underset{\uparrow}{\lambda} \nabla g$

Lagrange mult

Method:

1. Set up the system: $\begin{matrix} \nabla f = \lambda \nabla g \\ g = k \end{matrix}$ $(f, g \text{ in } \mathbb{R}^d)$
2. Solve for $(\vec{v}_1, \lambda_1), \dots, (\vec{v}_k, \lambda_l)$
3. Check values of $f(v_1), \dots, f(v_l)$
4. If $l \geq 2$, the max of those is max, min of those is min
   If $l = 1$, use context.

* Example

find min of $(x-1)^2+(y+1)^2$ subjected to $2x-y=6$

$$f(x,y) = (x-1)^2+(y+1)^2 \qquad g(x,y) = 2x-y$$

$$\nabla f = \langle 2(x-1), 2(y+1) \rangle \qquad \nabla g = \langle 2, -1 \rangle$$

Solve : $\nabla f = \lambda \nabla g$

$$2(x-1) = \lambda 2$$
$$2(y+1) = -\lambda$$
$$2x - y = 6$$

$\cdots \Rightarrow y = -\frac{8}{5}$, $x = -\frac{11}{5}$, $\lambda = \_$ ← Usually doesn't matter

So min is at $\left(-\frac{11}{5}, -\frac{8}{5}\right)$

find maximise volume of  subject to surface area $= 14$.

$$f(x,y,z) = xyz$$
$$g(x,y,z) = 2xy + 2yz + 2xz = 14$$

$$\nabla f = \langle yz, xz, xy \rangle \qquad \nabla g = \langle 2y+2z, 2x+2z, 2y+2x \rangle$$

$$yz = (2y+2z)\lambda$$
$$xz = (2x+2z)\lambda$$
$$xy = (2y+2x)\lambda$$
$$2xy + 2yz + 2xz = 14$$

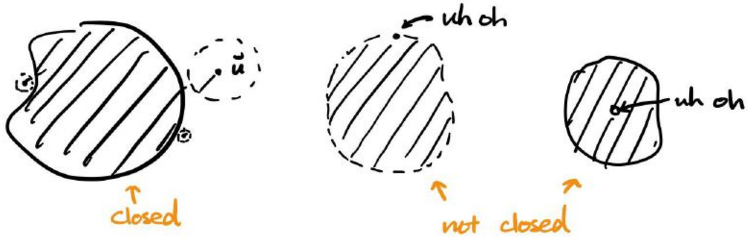
$\cdots \Rightarrow \frac{yz}{2(y+z)} = \frac{xy}{2(x+y)} = \frac{xz}{z(x+z)} \cdots \Rightarrow x = z, \ x = y, \ y = z$

$$\Downarrow$$
$$6x^2 = 14$$
$$x = \sqrt{\frac{7}{3}}$$

$\Rightarrow$ Crit point : $\left(\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$

Extreme value theorem

Def: A set $S \subseteq \mathbb{R}^d$ is closed if for each $\vec{v} \notin S$ $\exists \delta > 0$ s.t. all points $\vec{u} \in \mathbb{R}^d$ with $\text{dist}(\vec{v}, \vec{u}) < \delta$ are not in S



Thm: Level sets of continuous funcs are closed.

Ex. $\{(x,y) \mid x^2 - 2y^2 = 6\}$ is closed because it's level set of $x^2 - 2y^2$ and that's cont.

... to be continued