Lec 19 Double Integrals

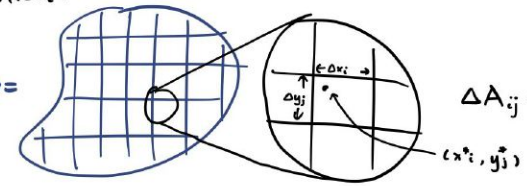
3D rectangular double integral

Let $z = f(x, y)$ be continuous on a closed, bounded domain $D \subseteq \mathbb{R}^2$.
A double integral is defed as:

$$\iint_D f \, dA = \lim_{n,m \to \infty} \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \, \Delta A_{ij}}_{\text{Riemann sum}}$$

Intuition:

D=



$\Delta A_{ij} = \Delta x_i \Delta y_i$

Slice → sample → sum

Compute by Iterated Integral

$z = xy - 2y^2 x$, $D = [0,1] \times [-1, 2]$

$$\iint_D xy - 2y^2 x \, dA = \int_{-1}^{2} \int_0^1 xy - 2y^2 x \, dx \, dy$$

$$= \int_{-1}^{2} \left(\frac{x^2 y}{2} - \frac{2y^2 x^2}{2} \Big|_{x=0}^{x=1} \right) dy$$

$$= \int_{-1}^{2} \frac{y}{2} - y^2 \, dy$$

$$= \frac{y^2}{4} - \frac{y^3}{3} \Big|_{-1}^{2}$$

← signed volume of this thing

Note: $\int_{-1}^{2} \int_0^1 xy - 2y^2 x \, dx \, dy$

$\shortparallel$

$\int_0^1 \int_{-1}^{2} xy - 2y^2 x \, dy \, dx$

↑

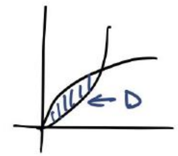
Fubini's theorem for rectangular region

If f continuous on $D = [a, b] \times [c, d]$, then

$$\iint_D f \, dA = \int_a^b \int_c^d f \, dy \, dx = \int_c^d \int_a^b f \, dx \, dy$$

Double integrals over general region & some defs

Ex: Let D be region between $y = x^2$ and $y = \sqrt{x}$



Eval: $\iint_D 1 \, dA = \int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} 1 \, dy \, dx$

generalises to more layers

pick one to scan over as outer, bounded by abs min/max of that var

the slice at that x, which is dependent on x, so generic expression for min/max of that var

whatever function

<u>Thm</u>: for close, bounded region $D \subseteq \mathbb{R}^2$, $\iint_D 1 \, dA$ is just the area of D.

<u>Def</u>: for cont f on close, bounded region $D \subseteq \mathbb{R}^2$, the average val of f on D is

$$f_{avg} = \frac{\iint_D f \, dA}{\text{Area}(D)}$$

Ex: find avg val of $z = 3xy$ between $y = x^2$ and $x = \sqrt{x}$

$$f_{avg} = \frac{\iint_D 3xy \, dA}{\frac{1}{3}} = 3 \int_0^1 \int_{x^2}^{\sqrt{x}} 3xy \, dy \, dx$$

(from previous) $= 3 \int_0^1 \left(\frac{3xy^2}{2} \Big|_{x^2}^{\sqrt{x}} \right) dx$

$\cdots = \frac{3}{4}$ (?)

Ex: find vol of tetrahedron with verts $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$