Lec 20

* Coord system + integral example

Describe region btwn $x^2+y^2=1$ and $x^2+y^2+z^2=4$ in cylindrical coord.

$$x^2+y^2=1 \;\downarrow\; r^2=1 \qquad\qquad x^2+y^2+z^2=4 \;\downarrow\; r^2+z^2=4$$

$\Rightarrow$ $1 \leq r \leq 2$

For fixed r, $-\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}$ ← Good for integration

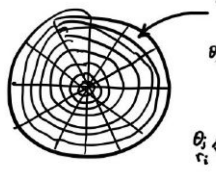
$0 \leq \theta \leq 2\pi$

We could also do:
$-\sqrt{3} \leq z \leq \sqrt{3}$
$1 \leq r \leq \sqrt{4-z^2}$
$0 \leq \theta \leq 2\pi$

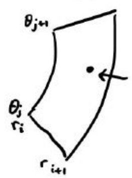
* $\iint$ in polar coord

Let f be cont. on disk D, do this to eval $\iint_D f\,dA$ in polar coord:

Riemann sum — another way to slice



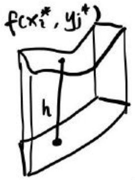
Polar rectangles $[\theta_j, \theta_{j+1}] \times [r_i, r_{i+1}] = A_{ij}$



sample point

$(x_i^*, y_j^*) = (r_i^* \cos\theta_j,\ r_i^* \cos\theta_j)$

Height at point: $f(r_i^* \cos\theta_j,\ r_i^* \cos\theta_j)$



Volume above rectangle $\approx f(r_i^* \cos\theta_j,\ r_i^* \cos\theta_j) \cdot \text{Area}(A_{ij})$

$= f(r_i^* \cos\theta_j,\ r_i^* \cos\theta_j) \cdot \frac{r_i + r_{i+1}}{2} \Delta\theta\Delta r$

Area(A_{ij})

$= \left(\frac{\theta_{j+1}-\theta_j}{2\pi}\right) \cdot (r_{i+1}^2\pi - r_i^2\pi)$

$= \frac{(r_{i+1}+r_i)}{2}(\theta_{j+1}-\theta_j)(r_{i+1}-r_i)$

Riemann sum: $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i^* \cos\theta_j,\ r_i^* \cos\theta_j) \cdot \underbrace{\frac{r_i + r_{i+1}}{2}}\Delta\theta\Delta r$

↳ for large enough n, this → r_i^*

Integral: $\displaystyle\iint_D f\,dA = \int_0^{2\pi}\int_0^R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$ ← aka the Jacobian of the polar change of var

Vol compute examples

Ex

region btwn $x^2+y^2=1$ and $x^2+y^2+z^2=4$

just a name for the region

We had: $E = \{ (r,\theta,z) \mid z^2 \le 3,\ 1 \le r \le \sqrt{4-z^2},\ 0 \le \theta \le 2\pi \}$

$$\mathrm{Vol}(E) = \iiint_E 1\, dV = \int_{-\sqrt{3}}^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} \int_0^{2\pi} r\, d\theta\, dr\, dz$$

Ex

Evaluate $\iint_R 3y\, dA$ for $R = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

$$= \int_1^2 \int_0^{\pi} 3r\sin\theta\ r\, dr\, d\theta$$

Doable geometrically or by integrating in (x,y,z) coord, but polar easier

Ex

Gaussian distribution !

Eval $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

Let $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$. then $I^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy = \int_0^{2\pi}\int_{-\infty}^{\infty} e^{-r^2/2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(-e^{-r^2/2} \Big|_0^{\infty} \right) d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi$$

So $I = \sqrt{2\pi}$