$\boxed{\text{Lec 22}}$

Change of variable to other coords in general

Recall: $\iint_D f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dA$

Ex. vol of 3D ball

$$Vol(E) = 2\iint_D \sqrt{R^2-x^2-y^2}\,dA \quad = 2\int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \sqrt{R^2-x^2-y^2}\,dx\,dy$$

polar $\rightarrow$ $$= 2\int_0^{2\pi}\int_0^{R} \sqrt{R^2-r^2}\; r\,dr\,d\theta$$

* Changing to cylindrical $\leftarrow$ similar to polar

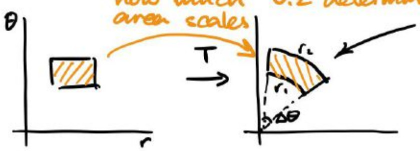
$$\underbrace{\iiint_E}_{\leftarrow \text{rect}} f(x,y,z)\,dV = \underbrace{\iiint_F}_{\leftarrow \text{cylindrical}} f(r\cos\theta, r\cos\theta, z)\; r\,dV$$

The Jacobian

* For polar, we have transformation:

how much area scales via determinant!

Area of this $= \frac{r_1+r_2}{2}\Delta r\,\Delta\theta$



$$T(r,\theta) \rightarrow (r\cos\theta, r\sin\theta)$$

* Jacobian can be computed by determinant!

Computing Jacobian

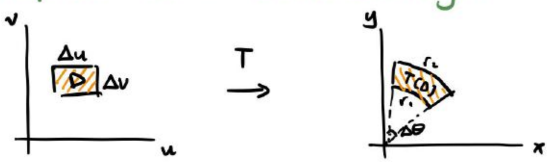
1. Let T be a 1 to 1 transformation from new coord to old coord

   Polar: $T(r,\theta) \to (r\cos\theta,\ r\sin\theta)$

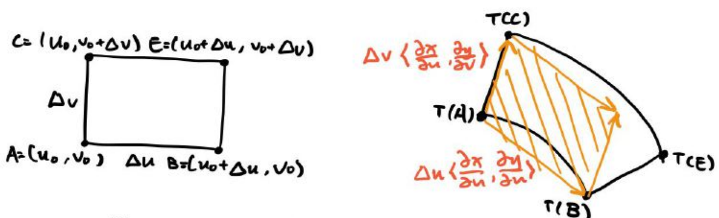
   Cylin: $T(r,\theta,z) \to (r\cos\theta,\ r\sin\theta,\ z)$

   Spheri: $T(\rho,\theta,\varphi) \to (\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\theta)$

2. Transform some small rectangle



$$T(u,v) = \langle \underbrace{x(u,v),\ y(u,v)}_{\text{can be generic}} \rangle$$



For small enough D:

Area $(T(D))$

$\approx$ area of parallelagram determined by $\vec{AC}$ and $\vec{AB}$

$= \| \vec{AC} \times \vec{AB} \|$

$= \| \langle T(u_0+\Delta v, v_0) - T(u_0,v_0) \rangle \times \langle T(u_0, v_0+\Delta v) - T(u_0,v_0) \rangle \|$

$= \| \Delta u \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \rangle \times \Delta v \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \rangle \|$

$$= \Delta u \Delta v \underbrace{\left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} \right|}_{\text{Jacobian}}$$

3D Jacobian, very similar

Let $T(u,v,w) \to (x,y,z)$ be a 1 to 1 transformation

Jacobian of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

<u>Thm</u>: If T is also smooth,

$$\iiint_E f(x,y,z)\,dV = \iiint_{T^{-1}(E)} f(T(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV$$

Ex: rect to cylin. $T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Ex. rect to spheri

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} \cdots = \rho^2 \sin\varphi$$