$\boxed{\text{Lec 23}}$

Change of var — higher dim u-sub

Ex (recall u-sub)

$$\int_1^2 x(x^2+1)\,dx = \int_2^5 \tfrac{1}{2}u\,du \qquad \begin{matrix} u = x^2+1 \\ du = 2x\,dx \end{matrix} \qquad \begin{matrix} x = \sqrt{u-1} \\ dx = \frac{du}{dx} \end{matrix}$$

For $\iint$

$\iint_D f(x,y)\,dA$ f cont, D closed & bounded, $D \subseteq \mathbb{R}^2$

want $T : \underset{\substack{\uparrow \\ \text{in } (u,v) \text{ coord}}}{S} \to D$ ↖ in (x,y) coord

want T being injective i.e. $T(u_1,v_1) = T(u_2,v_2) \Leftrightarrow (u_1,v_1) = (u_2,v_2)$

then $\exists\, T^{-1} : D \to S$

also want T to be $\underline{C^1}$ transformation $T(u,v) = (x(u,v), y(u,v))$

└ viz $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ all exist and cont.

Thm

$$\iint_D f(x,y)\,dA = \iint_{S = T^{-1}(D)} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

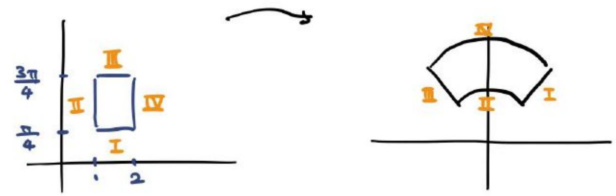
Ex. change to polar coord

$T(r,\theta) = (r\cos\theta, r\sin\theta)$ for $0 \le \theta < 2\pi$, $r > 0$, which is C^1

Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} =$

Given $S \subseteq \mathbb{R}^2$ in (u,v), finding $T(S)$

Ex. $S = \{(r,\theta) \mid \frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\ 1 \le r \le 2\}$



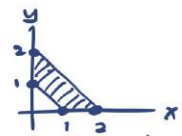
Check boundary !

I : $\{(r,\theta) \mid \theta = \frac{\pi}{2},\ 1 \le r \le 2\}$

$T(I) = \{(r\cos\theta, r\sin\theta) \mid \theta = \frac{\pi}{4},\ 1 \le r \le 2\}$

II III IV [omit]

Ex. $\iint_D (x^3 + 3x^2y + 3xy^2 + y^3)\, dA \qquad D =$

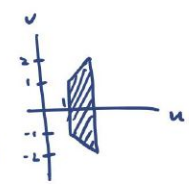


using $u = x+y$, $v = x-y$

$T^{-1} : (x,y) \to (x+y,\ x-y)$

Solve: $u = x+y$, $v = x-y$ $\Rightarrow$ $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

$\hookrightarrow T(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$



Then $f(x,y) = f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdots = u^3$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

$$\iint_D x^3 + 3x^2y + 3y^2x + y^3\, dA = \iint_{T^{-1}(D)} u^3 \cdot |-1/2|\, dA$$

partial deri all constant

* When T (or T^{-1}) is linear, we can just check vertices

$T^{-1}(0,1) = (1,-1) \quad T^{-1}(0,2) = (2,-2) \quad T^{-1}(1,0) = (1,1) \quad T^{-1}(2,0) = (2,2)$

$$\hookrightarrow\ = \int_1^2 \int_{-u}^{u} u^3\, |-\tfrac{1}{2}|\, du\, dv$$