$\boxed{\text{Lec 24}}$ Apps of integrals in probability and centre of mass

Density

Def A <u>density function</u> ρ on $D \subseteq \mathbb{R}^d$ is a func $\rho(\vec{x}) \geq$ for all $\vec{x} \in D$ so that

$$\underbrace{\iint \cdots \int \rho \, dV}_{\text{the mass}} \in (0, \infty)$$

Def A <u>probability density</u> on D is density ρ s.t. the mass $= 1$

Ex. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, which satisfies $\int_{-\infty}^{\infty} f \, dx = 1$

Aka Gaussian / normal distribution

Def <u>Continuous random variable</u> is what we get if we sample X from some ρ

Ex. $X \sim N(0,1)$ — X sampled from standard normal

$\Pr(X \leq x_0) = \int_{-\infty}^{x_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

Ex. $X \sim \operatorname{Exp}(\lambda)$ — $f(x) = \lambda e^{-\lambda x}$ on $[0, \infty)$ — exponential distribution, with param $\lambda > 0$

$\Pr(X \leq x_0) = \int_0^{x_0} \lambda e^{-\lambda x} dx$

$\mathbb{E}(X) = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$

Def <u>Expected val</u> or <u>expectation</u> or <u>average val</u> of X is

$$\mathbb{E}(X) = \int \cdots \int_D x \rho \, dV$$

Multivariable — density in ≥ 2 variables, sample vector

Ex. select (X, Y) as a point uniformly from unit disk at origin

$\rho(x, y) = \frac{1}{\pi}$ ← <u>Uniform distribution</u> : density is always $\frac{1}{\text{volume}}$

$\iint_D \rho \, dA = 1$

* for cont. rand. vec. (X,Y) on D and $E \subseteq D$,
$\Pr((X,Y) \in E) = \int \cdots \int_E \rho \, dV$

Same Ex, but what is $p(X^2+Y^2 \leq R^2)$ for $R \leq 1$

$$\Pr(X^2+Y^2 \leq R^2) = \iint_E \rho \, dA = \iint_E \frac{1}{\pi} \, dA = \frac{\text{Area}(E)}{\pi} = R^2$$

<u>Def</u> Two rand. vars X and Y are indep if $\rho_{X,Y} = \rho_X \rho_Y$

↪ Alt: $\Pr(X \leq x \wedge Y \leq y) = \Pr(X \leq x)\Pr(Y \leq y)$

Ex. $X \sim \text{Exp}(1)$ $Y \sim \text{Exp}(2)$ are indep

$$\rho_{X,Y} = \rho_X \rho_Y = e^{-x} \, 2e^{-2y} = 2e^{-x-2y}$$

$$\Pr(X+Y \leq 1) = \iint_E 2e^{-x-2y} \, dA$$