Lec 26

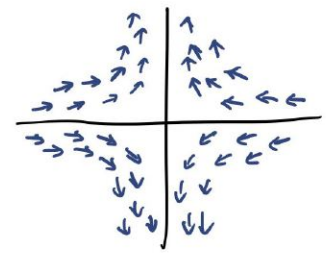
Vector Field (V.F.)

Def: a func $\vec{F}: D \to \mathbb{R}^n$ for $D \subseteq \mathbb{R}^n$ in the form

$$\vec{F} = \langle f, \dots, f_n \rangle$$

in which each $f_i: D \to \mathbb{R}$ is a component func

$\vec{F} = \langle -\frac{x}{2}, \frac{y}{2} \rangle$



Ex. $\vec{F}(x,y) = \langle 2xy, x^2y^2 \rangle$ in $\mathbb{R}^2$

Notation Usually we use P, Q, R for component func

$\vec{F} = \langle P, Q \rangle$ in $\mathbb{R}^2$, $\vec{F} = \langle P, Q, R \rangle$ in $\mathbb{R}^3$

Classes - Gradient vec field

$f: \mathbb{R}^n \to \mathbb{R}$ ← scalar field

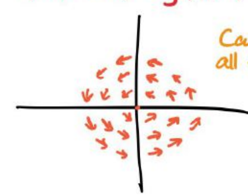
$\nabla f = \langle f_x, f_y, f_z \rangle$ ← an $\mathbb{R}^3$ vec field

Def A vec field $\vec{F}$ is a <u>conservative V.F.</u> aka <u>gradient V.F.</u> if $\vec{F} = \nabla f$ for a scalar func f.

Ex. $\vec{F} = \langle 2x, -2y \rangle$ conservative since $\nabla(x^2 - y^2) = \vec{F}$

Non Ex. $\vec{F}? = \langle -y, x \rangle$ ← rotational V.F. not conservative.

$\vec{F}? = \langle -y, x \rangle$



Can't increase all the time

Thm For $\vec{F}(P,Q)$, if P, Q are C^1 and $P_y \neq Q_x$, then $\vec{F}$ not conservative

Not an iff !

Def That f is called a potential function

Ex. (cont) $f = x^2 + y^2 + C$

↳

<u>Proof</u> If $\vec{F} = \langle P, Q \rangle$ conservative, $\exists f$ s.t. $f_x = P$, $f_y = Q$.
Since P, Q are c^1, f is c^2. Then by Clairaut's thm
$f_{xy} = f_{yx} \Rightarrow P_y = Q_x$

Consider $\vec{F} = \left\langle \underbrace{\frac{-y}{\sqrt{x^2+y^2}}}_{P}, \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{Q} \right\rangle$

Will find $P_x = Q_y$, but $\vec{F}$ is still non conservative

* In general, $P_y = Q_x$ but $\langle P, Q \rangle$ not conservative always has to do with hole in domain

<u>Thm</u> If $\vec{F}$ defed on a $\overbrace{\text{simply connected}}^{\text{viz domain } D \text{ doesn't have hole}}$ $D \subseteq \mathbb{R}^2$, then
$$\vec{F} \text{ conservative} \Leftrightarrow P_y = Q_x$$