$\boxed{\text{Lec 27}}$

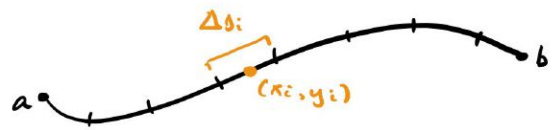
\# Line integral of scalar field

Def Let $z = f(x, y)$ be cont. curve C in $\mathbb{R}^2$ parametrised by $\vec{r}(t) = \langle x(t), y(t) \rangle$ for $a \le t \le b$, f is C^2

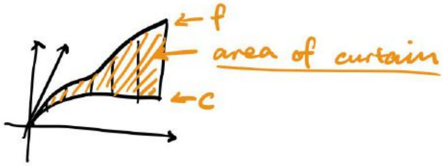
The <u>line integral</u> over C is defined as:

$$\int_C f\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta s_i$$

arc length



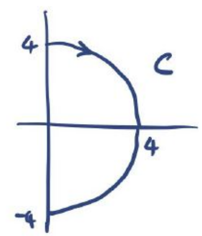
<u>Geometric interp</u>



Thm Observe that $ds = \| r'(t) \|\, dt$. So:

$$\int_C f\, ds = \int_a^b f(\vec{r}(t))\, \| r'(t) \|\, dt$$

Ex. eval line integral of xy^4 over that C in that direction



$\vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$ from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$

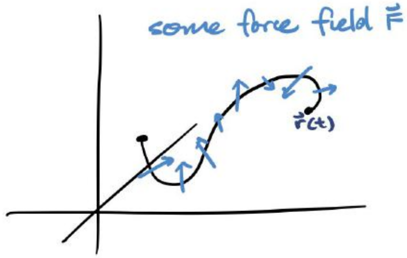
$\vec{r}'(t) = \langle -4\sin t, 4\cos t \rangle$

$\| \vec{r}'(t) \| = 4$

$$\int_C xy^4\, ds = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} 4\cos t\, (4\sin t)^4\, 4\, dt \cdots = 4096\, \frac{\sin^5 t}{5}$$

Line integral of vector field

Motivation: suppose $\vec{F}$ is a force vec and $\vec{d}$ is a displacement vec. Then work $W = \vec{F}\cdot\vec{d}$
but what if $\vec{F}$ and $\vec{d}$ both change depending on x, y
viz. $\vec{F} = \langle P(x,y), Q(x,y) \rangle$
$\vec{r}(t) = \langle x(t), y(t) \rangle$



Work approximation at segment

$\vec{F}\cdot\vec{T}(x_i^*, y_i^*)\,\Delta s_i$

$\Rightarrow W \approx \sum_{i=1}^{n} \vec{F}\cdot\vec{T}(x_i^*, y_i^*)\,\Delta s_i$

send n to ∞ and we get.–

Recall: $\vec{T}(t) = \frac{r'(t)}{\|r'(t)\|}$

Def line integral of $\vec{F} = \langle P, Q \rangle$ along C is

standard notation

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\frac{r'(t)}{\|r'(t)\|}\|r'(t)\|\,dt = \int_C \vec{F}\cdot r'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}$$

Also, if $\vec{F} = \langle P, Q \rangle$,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

Thm If C is c', $\vec{F} = \langle P, Q \rangle$ cont, P, Q cont,

$$\int_C \vec{F}\cdot dr = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

Ex. $\int_C yz\,dx + xz\,dy + xy\,dz$ and $\vec{r}(t) = \langle 1+2t, 1+t, 1-t \rangle$, $0 \le t \le 1$

$= \int_C \langle yz, xz, xy \rangle \cdot \langle 2, 1, -1 \rangle\,ds$