Lec 28

* More conser. V.F.

Recall V.F. $\vec{F}$ conservative if $\vec{F} = \nabla f$ for some potential func f

Recall $\vec{F}\langle P, Q\rangle$ conser. $\Rightarrow P_y = Q_x$; $P_y \neq Q_x \Rightarrow \vec{F}$ not conser. ← cross partial prop

<u>Finding</u> the potential func

Ex. $\vec{F} = \langle 2xy - 2y + (y+1)^2,\ (x-1)^2 + 2xy + 2x \rangle = \langle P, Q \rangle$ (indeed $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in this example)

Antiderivative !

~~WTF~~ ? Want to find: $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$

$f = \int 2xy - 2y + (y+1)^2 \, dx$ ← should match → $f = \int (x-1)^2 + 2xy + 2x \, dy$

$= x^2y - 2yx + (y+1)^2 x + g(y)$ ← some func on y, which in constant w.r.t. x

$= (x-1)^2 y + xy^2 + 2xy + h(x)$

$= x^2y - 2xy + y^2x + 2xy + x + g(y)$ ↔ only need to check terms with both vars agree

$= x^2y - 2xy + y + xy^2 + 2xy + h(x)$

$f = x^2y - 2xy + y^2x + 2xy + x + y + C$ Found f, so $\vec{F}$ conser.

Non-Ex. $\vec{F} = \langle -y, x \rangle$

$f = \int -y \, dx = -xy + g(y)$ $f = \int x \, dy = xy + h(x)$

Not match

↓

$\vec{F}$ not conser.

Fundamental Thm of Line Integral (FTLI)

<u>Thm</u> If $\vec{F}$ is c^2 func and C is piecewise smooth curve $\vec{r}(t)$ for t from a to b, then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

<u>Proof</u>

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f((r(t)) \cdot r'(t)\, dt$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}(\vec{r}(t)), \frac{\partial f}{\partial y}(\vec{r}(t)) \right\rangle \cdot \left\langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right\rangle dt$$

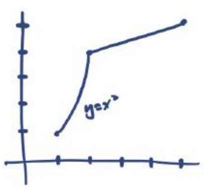
$$= \int_a^b \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\, dt$$

$$= \int_a^b \frac{\partial f}{\partial t}\, dt$$

$$= \int_a^b f(\vec{r}(t))\, dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

Ex. Let C be



Find $\int_C \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2xy + 2x \rangle \cdot d\vec{r}$

$$= x^2y - 2xy + y^2x + 2xy + x + y + C \Big|_{(1,1)}^{(5,5)}$$

$$= 256$$

<u>Coro</u> If $\vec{F}$ conser., C_1, C_2 piecewise smooth with same endpoints,

$$\int_{C_1} F \cdot d\vec{r} = \int_{C_2} F \cdot d\vec{r}$$

* One more theorem

<u>Thm</u> If $\vec{F} = \langle P, Q \rangle$ and domain of $\vec{F}$ simply-connected,

$P_y = Q_x \iff f$ conser.

path connected & no hole.

yes

$\mathbb{R}^2$

no

$\mathbb{R}^2 \setminus \{(0,0)\}$