Lec 29

Green's Theorem

<u>Notation</u> Let c be piecewise smooth curve, simple, closed curve in $\mathbb{R}^2$,

$$\oint_c \vec{F}\cdot dr = \int_c \vec{F}\cdot dr \quad \text{in counterclockwise direction.}$$

closed = start and end at some point
simple = no self intersection


closed simple


closed not simple

<u>Notice</u> If $\vec{F}$ cons,

$$\oint_c \vec{F}\cdot dr = 0 \quad \text{by FTLI}$$

<u>Thm</u> $\vec{F} = \langle P, Q\rangle$ in simply conn. region D in $\mathbb{R}^2$, then

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

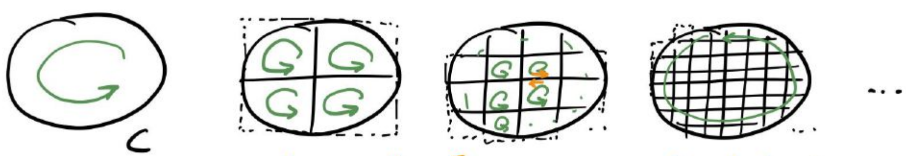
← boundary of D

Ex. $\vec{F} = \langle -y, x\rangle$, C be unit disk

$$\oint_C \vec{F}\,dr = \iint_D (Q_x, P_y)\,dA = \iint_D (1-(-1))\,dA = 2\iint_D dA = 2\pi$$

<u>Proof</u> (special case sketch)

Given $\vec{F} = \langle P, Q\rangle$, $Q_x - P_y$ measures rotation of vec field around point



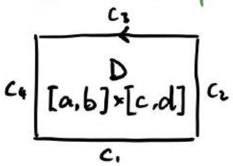
WT measure

Break into small pieces

Inner ones cancel out

Approximate outer rotation

Now we prove Green's Thm for rectangle



$F = \langle P, Q \rangle$

WTS $\oint_c P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$

$c = c_1 + c_2 + c_3 + c_4$

$$\oint_c P\,dx + Q\,dy = \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4} P\,dx + Q\,dy$$

$c_1 \quad r_1(t) = \langle t, c \rangle$

$c_2 \quad r_2(t) = \langle b, t \rangle$

$-c_3 \quad r_3(t) = \langle t, d \rangle$

$-c_4 \quad r_4(t) = \langle a, t \rangle$

$$\text{LHS} = \int_a^b \langle P(t,c), Q(t,c) \rangle \cdot r_1'(t)\,dt + \int_c^d \langle P(b,t), Q(b,t) \rangle \cdot r_2'(t)\,dt +$$

$$\int_b^a \langle P(t,d), Q(t,d) \rangle \cdot r_3'(t)\,dt + \int_d^c \langle P(a,t), Q(a,t) \rangle \cdot r_4'(t)\,dt$$

$$= \int_a^b \langle P(t,c), Q(t,c) \rangle \cdot \langle 1,0 \rangle\,dt + \int_c^d \langle P(b,t), Q(b,t) \rangle \cdot \langle 0,1 \rangle\,dt +$$

$$\int_b^a \langle P(t,d), Q(t,d) \rangle \cdot \langle 1,0 \rangle\,dt + \int_d^c \langle P(a,t), Q(a,t) \rangle \cdot \langle 0,1 \rangle\,dt$$

$$= \int_a^b P(t,c)\,dt + \int_c^d Q(t,d)\,dt + \int_b^a P(b,t)\,dt + \int_d^c Q(a,t)\,dt$$

$$= \int_a^b P(t,c) - P(b,t)\,dt + \int_c^d Q(t,d) - Q(a,t)\,dt$$

$$\text{RHS} = \iint_D Q_x - P_y\,dA = \int_a^b \int_c^d (Q_x - P_y)\,dx\,dy$$

$$= \int_a^b \int_c^d Q_x\,dx\,dy - \int_a^b \int_c^d P_y\,dx\,dy$$

by FTC

$$= \int_a^b Q(b,y) - Q(a,y)\,dy - \int_a^b P(x,d) - P(x,c)\,dx = \text{LHS}$$

Flux version Green's Thm

normal vec

Def $\oint_C \vec{F}\cdot\vec{N}\,ds$ is the flux integral $\int_C \vec{F}\cdot\vec{N}\,ds$ in counterclockwise direction.

measures net outflow in region enclosed by C

measures flow across C
(recall line integral $\int_C \vec{F}\cdot\vec{T}\,ds$ measures flow along C)

Thm $\oint_{\partial D}\langle P, Q\rangle\cdot\vec{N}\,ds = \iint_D \underbrace{(P_x + Q_y)}_{\text{outflow at each point}}\,dA$