Lec 30

Curl & Divergence

<u>Def</u> For vec field $\vec{F} = \langle P, Q \rangle$ in $\mathbb{R}^2$, the <u>curl</u> of $\vec{F}$ is

$$\text{curl } \vec{F} = \langle 0, 0, Q_x - P_y \rangle$$ ← net rotation around point

and the divergence

$$\text{div } \vec{F} = P_x + Q_y$$ ← net outflow at point

<u>Notice</u>

$$\oint_{\partial D} \vec{F} \cdot dr = \iint_D Q_x - P_y \, dA = \iint_D \text{curl } \vec{F} \cdot \hat{k} \, dA$$

$$\oint_{\partial D} \vec{F} \cdot \vec{N} \, ds = \iint_D P_x + Q_y \, dA = \iint_D \text{div } \vec{F} \, dA$$

Extended Green's Theorem

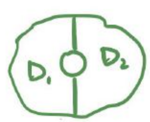
<u>Thm</u> We can replace the assumption that D is simply connected with the assumption that D has finitely many holes
Viz. both hold with new assumption

<u>Derivation</u> of ∂D by example

Suppose:

not simply connected

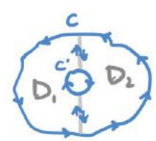
We can:



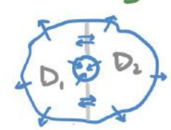
two simply connected

Then do both on D_1 and D_2. Notice boundry cancels out

Rotation



Flow



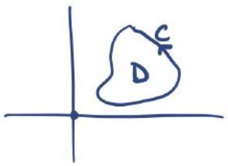
Then $\partial D = C - C'$

$$\oint_{\partial D} F \cdot d\vec{r} = \oint_C F \cdot d\vec{r} - \oint_{C'} F \cdot d\vec{r}$$

Ex. show $F = \langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \rangle$ on any simple closed C that doesn't contain origin satisfies

$$\oint_C \vec{F} \cdot dr = \begin{cases} 0 & \text{if } C \text{ doesn't surround } (0,0) \\ -2\pi & \text{else} \end{cases}$$

Case ①: if C doesn't surround $(0,0)$



By ordinary Green's

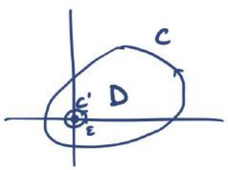
$$\oint_c F \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$$

$$= \iint_D \frac{x^2-y^2}{x^2+y^2} - \frac{x^2-y^2}{x^2+y^2} \, dA$$

$$= \iint_D 0 \, dA$$

$$= 0$$

Case ②: if C contains origin



Let C' be a circle inside C around origin viz. radius ε.

Let D be inside C outside C'.

By extended Green's

$$\oint_{\partial D} F \cdot d\vec{r} = \iint_C \vec{F} \cdot d\vec{r} - \iint_{C'} \vec{F} \cdot d\vec{r} = 0$$ ← since D doesn't contain origin

$$\Rightarrow \iint_C \vec{F} \cdot d\vec{r} = \iint_{C'} \vec{F} \cdot d\vec{r}$$ ← easier

Param C as $\vec{r}(t) = \langle \cos t, \sin t \rangle$ ← scaling doesn't matter since things same

$$\oint_c \vec{F} \cdot dr = \iint_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot r'(t) \, dt$$

$$\cdots = -2\pi$$