Lec 31

3D divergence

<u>Notice</u> Divergence, curl, gradient all send some scalar func / scalar field to some scalar func / scalar field

Let $\vec{F} = \langle P, Q, R \rangle$ (or $\langle P, Q \rangle$)

<u>Def</u> $\operatorname{div} \vec{F} = P_x + Q_y + R_z = \nabla \cdot \vec{F}$ ← measures tendency for vecs to flow out of point

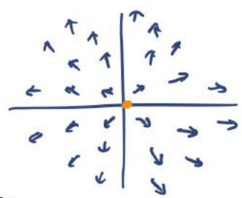
<u>Notation</u> $\nabla \cdot \vec{F}$ is dot prod of operator — apply derivative to each entry viz.

$$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \left\langle \frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z} \right\rangle$$

Ex. $\vec{F} = \left\langle \frac{x}{x^2+}, \frac{y}{x^2+y^2} \right\rangle$

$\operatorname{div} \vec{F} = \frac{y^2 - x^2}{(x^2+y^2)} + \frac{x^2 - y^2}{(x^2+y^2)}$

$= 0$ everywhere other than $(0,0)$



← radial field

Everything sourced here...

<u>Intuition</u> $\oint_{\partial D} \vec{F} \cdot \vec{N} \, ds = \iint_D \nabla \cdot \vec{F} \, dA$ ← add up net outflow of all points gives net outflow across region

<u>Def</u> A vec field in $\mathbb{R}^2$ is <u>source free</u> or <u>magnetic</u> if for all $D \subseteq \mathbb{R}^2$,

$$\oint_{\partial D} \vec{F} \cdot \vec{N} \, ds = 0$$

<u>Thm</u> If $\vec{F}$ source free, then $\operatorname{div} \vec{F} = 0$ on domain of $\vec{F}$

If domain $\vec{F}$ simply connected and $\operatorname{div} \vec{F} = 0$, then $\vec{F}$ source free

Fund. operations in vec calc

gradient ∇ : scalar field → vec field

divergence div : vec field → scalar field

curl curl : vec field → vec field

3D curl

Def curl $\vec{F} = \nabla \times \vec{F}$ ← measures flow around a point

viz.

$$\text{curl } \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z , P_z - R_x , Q_x - P_y \rangle$$

In $\mathbb{R}^2$, curl $\vec{F}$ points in dir of axis of rotation, and $\|\text{curl } \vec{F}\|$ measures speed

Circulation Green's

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \vec{k} \, dA$$

Thm If $\vec{F}$ conservative, then curl $\vec{F} \equiv \vec{0}$ ← (restatement)

If domain $\vec{F}$ simply connected, curl $\vec{F} \equiv 0 \Rightarrow \vec{F}$ conservative