Lec 32 Parametrised Surface & Surface Integral

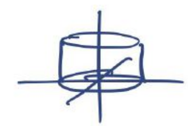
→ Net outflow through surface in $\mathbb{R}^3$ or even higher dim?

Parametrised curve

Recall parametrised curve $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

But we can make $\vec{r}$ take two params to get surf

Ex. $\vec{r}(u,v) = \langle \cos u, \sin u, v \rangle$ for $0 \le u \le 2\pi$, $0 \le v \le 1$



Ex. $\vec{r}(\theta,\varphi) = \langle \sin\varphi \cos\theta, \sin\varphi \cos\theta, \cos\varphi \rangle$ for $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$

Ex. parametrise $ax + by + cz = d$

→ Solve for z: $z = \frac{1}{c}(d - ax - by)$

⇒ $\vec{r}(x,y) = \langle x, y, \frac{1}{c}(d - ax - by) \rangle$

Ex. $\vec{r}(u,v) = \langle u, 0, v, \sqrt{1 - v^2 - u^2} \rangle$ ← Surface in 4D

Surface Integral

Assume: · S is our smooth, parametrised surf by $\vec{r}(u,v)$ for $(u,v) \in D$

↳ $\vec{r}_u, \vec{r}_v$ both defed and $\vec{r}_u \times \vec{r}_v$ is never 0.

↳ else we don't get surface change

Surface area

$$\iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA$$

each ▱ is $\vec{r}(u,v)$ for $(u,v) \in [a_i, a_{i+1}] \times [b_j, b_{i+1}]$. Area $\|\vec{r}_u \times \vec{r}_v\| \Delta a \Delta b$.

Def Given cont funct $f(x,y,z)$ in $\mathbb{R}^3$, the surf integral is

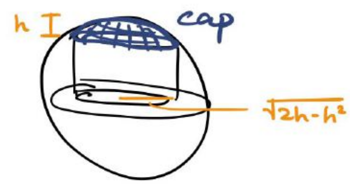
$$\iint_S f\,dS = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\vec{r}(u^*, v^*))\,(\text{Area of } S \text{ on } [u_i, u_{i+1}] \times [v_j, v_{j+1}])$$

In practice

$$\iint_S f\,dS = \iint_D f(\vec{r}(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,dA$$

(Note if $f=1$ then this gives surface area)

Ex. surf area of a spherical cap on a unit sphere



Parametrize : $\vec{r}(x,y) = \langle x, y, \sqrt{1-x^2-y^2} \rangle$

Domain : $D = \{(x,y) \mid x^2+y^2 \le 2h-h^2\}$

Area:

$$\iint_D \|r_x \times r_y\|\,dA$$

$$= \iint_D \left\|\left\langle 1, 0, \frac{-x}{\sqrt{1-x^2-y^2}} \right\rangle \times \left\langle 0, 1, \frac{-y}{\sqrt{1-x^2-y^2}} \right\rangle\right\| dA$$

$$\cdots = \iint_D \left\|\left\langle \frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1 \right\rangle\right\| dA$$

$$\cdots = 2\pi h$$